

PART I

QUANTUM MECHANICS

I. Consider a one-dimensional Schrödinger equation in x-space,

$$\left[\frac{d^2}{dx^2} - v(x) + k^2 \right] \psi_k(x) = 0 \quad (1)$$

where

$$v(x) = \frac{2m}{\hbar^2} V(x). \quad (1a)$$

A.) Show that the Green's function for the one-dimensional Helmholtz equation is given by

$$G_k(x | x') = - \frac{e^{ik|x-x'|}}{2ik} \quad (2)$$

B.) Show that the integral equation equivalent to Eq. (1) with the boundary condition of the incident plane wave coming from left is given by

$$\begin{aligned} \psi_k(x) = & \frac{e^{ikx}}{\sqrt{2\pi}} + \frac{e^{ikx}}{2ik} \int_{-\infty}^x e^{-ikx'} v(x') \psi_k(x') dx' \\ & + \frac{e^{-ikx}}{2ik} \int_x^{\infty} e^{ikx'} v(x') \psi_k(x') dx'. \end{aligned} \quad (3)$$

C.) Henceforth assume the following potential:

$$v(x) = -\gamma \delta(x) \quad (4)$$

where

$$\gamma = \frac{2m}{\hbar^2} V_0 \quad (4a)$$

Show that Eq.(4) then admits the following normalized bound state solution

$$\psi_b(x) = \sqrt{\frac{\gamma}{2}} e^{-\gamma|x|/2} \quad (5)$$

with the binding energy

$$E_b = -\frac{\hbar^2 \gamma^2}{8m} \quad (5a)$$

- D.) By employing the potential (4), solve the scattering problem of (B). Show that

$$\Psi_k(x) = \frac{1}{\sqrt{2\pi}} \left[e^{ikx} + R(k) e^{-ikx} \theta(-x) + S(k) e^{ikx} \theta(x) \right] \quad (6)$$

where

$$\begin{aligned} \theta(x) &= 1 & x > 0 \\ &= 0 & x < 0, \end{aligned}$$

and

$$R(k) = \frac{i\tau}{2k - i\tau} = S(k); \quad T(k) \equiv S(k) + 1 \quad (7)$$

Discuss the flux conservation as well as the physical meaning of the coefficients of $R(k)$, $S(k)$ and $T(k)$.

- E.) The bound state solution $\Psi_b(x)$ of (5) can be obtained also from an integral equation. How should Eq.(3) be modified? By examining the analytic properties of $S(k)$ and $R(k)$ in the complex k -plane one can evaluate $\Psi_b(x)$ as well as E_b . How would you go about this? What is the value of k that yields the bound state? How should G_k of (2) be modified for the bound state problem?

- F.) Convert Eq.(1) into the momentum (p)-space. Show that the one-dimensional Schrödinger equation in p -space reads

$$(p^2 - k^2) \varphi_k(p) = \int_{-\infty}^{\infty} \frac{dq}{\sqrt{2\pi}} \bar{v}(p-q) \varphi_k(q). \quad (8)$$

Here

$$\bar{v}(p) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} v(x) e^{-ipx} \quad (8a)$$

where $v(x)$ is the local potential in Eq.(1a).

QUANTUM MECHANICS--Cont.

- G.) Employ the δ -potential, Eq.(4), for $v(x)$. Show that p-space scattering solution that satisfies the outgoing wave boundary condition is given by

$$\psi_k(p) = \delta(k-p) - \frac{\gamma}{2\pi} \frac{1}{D(k)} \frac{1}{k^2 - p^2 + i\epsilon} \quad (9)$$

where

$$D(k) = 1 + \frac{\gamma}{2\pi} \int_{-\infty}^{\infty} \frac{dq}{k^2 - q^2 + i\epsilon} \quad (9a)$$

Verify that the scattering and reflection coefficients are given by (7).

- H.) Show that the bound state solution $\psi_b(x)$ of (5) can also be obtained by working on Eq.(8).
 (Hint: To simplify the treatment, show that $\psi_k(p)$, the Fourier transform of $\psi_b(x)$, satisfies Eq.(8).)

PART 2

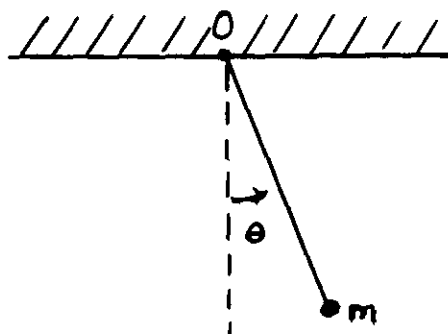
MECHANICS

- I. A.) Consider a simple pendulum formed by a point weight on a string suspended from the point C on the ceiling.

Write down the Hamiltonian of the system. Solve the Hamilton-Jacobi equation for small amplitude of oscillations and show that the energy is given by

$$E = J\nu$$

where J is the action, and ν the frequency.



- B.) Suppose that the length of the pendulum is decreased by slowly pulling the string up through the hole located at C.

Solve Hamilton's equation of motion for small values of θ for a fixed value of the string length. Compute the change in the energy of the pendulum, as the length is shortened, from the work done in pulling against the tension of the string. Show that, as a result of the change of the string length at a rate slow compared to the intrinsic frequency of the system, the action variable $J = E/\nu$ remains constant throughout the process.

This is the principle of adiabatic invariance. State the principle succinctly in your own words.

- C.) Discuss qualitatively the quantum mechanical description of low energy molecular elastic collisions by invoking the principle of adiabatic invariance. Draw upon a close analogy between this problem and the physics of the pendulum discussed in (B.).

Miscellaneous topics

PART 3

~~general~~ topics

~~METHODS OF THEORETICAL PHYSICS~~

I) Assume that in a block of uranium 235 the neutron density $n(\vec{x}, t)$ obeys the differential equation

$$\nabla^2 n + \lambda n = \frac{1}{\mu} \frac{\partial n}{\partial t}$$

and that $n=0$ at the surface. It is desired to put the material in a cylindrical piece. Find the minimum dimensions of the cylinder which becomes supercritical (i.e., the neutrons regenerate without bound in time.)

II) For a linear system the input $f(t)$ and the output $F(t)$ are linked by a causal (or Green's) function as follows:

$$F(t) = \int G(t - t') f(t') dt' .$$

If the input has the particular form $f(t) = \theta(t) e^{-\lambda t}$

$$\text{where } \theta(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

the output is observed to be

$$F(t) = \theta(t) (1 - e^{-\alpha t}) e^{-\lambda t} .$$

Find the "response" function $G(\omega)$, the Fourier transform of $G(\tau)$, and show that it is analytic in the upper half complex ω plane. Find the causal function $G(\tau)$ and show that

$$G(\tau) = \begin{cases} 0 & \tau < 0 \\ \neq 0 (?) & \tau > 0 \end{cases} .$$

Interpret the above result in physical terms and its bearing on the analytic properties of $G(\omega)$ (Titchmarsh theorem).

PART III

Ph. D. Qualifying Examination

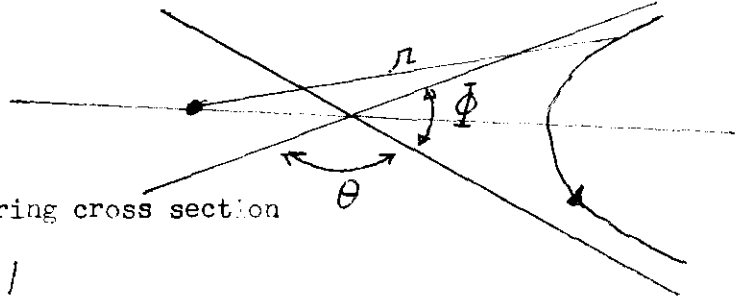
June 1967

III

Rutherford

A) Derive the equations of motion for the ~~st~~ ~~xxx~~ scattering of alpha particles by atomic nuclei the following methods:

1. Simple Newtonian mechanics
2. Lagrange's equations
3. Hamilton's equations
4. Hamilton-Jacobi method.



B) Derive the famous Rutherford scattering cross section

small e

$$\sigma(\theta) = \frac{1}{4} \left(\frac{ZZ'e^2}{2E} \right)^2 \frac{1}{\sin^4 \frac{\theta}{2}}$$

C)

~~Derive~~ In atomic physics the ~~concept~~ of total scattering cross section is given by

$$\sigma_t = \int_{4\pi} \sigma(\Omega) d\Omega = 2\pi \int_0^\pi \sigma(\theta) \sin \theta d\theta$$

Substitute the value of the scattering cross section given in Question II in this equation and discuss the ~~xxx~~ results.

IV. Transform the scattering problem to laboratory coordinates.

V. Find the principal moments of inertia about the center of mass of a flat rigid body in the shape of a 45° right triangle with uniform mass density. Designate the principal axes.

June 13, 1967

Tuesday

Part ~~IV~~, ~~V~~, VI

General Instructions

- 1.) Solve at least one problem in each of part 4, 5, and 6
- 2.) You may use the tables of integral provided to you.
- 3.) Time schedule for the test:
9:00AM - 12:00 AM
1:00PM - 4:00 PM

- I. A.) Show that, if the index of refraction $n(\omega)$ is analytic in the upper half complex ω -plane and approaches unity for large $|\omega|$, its real and imaginary parts are related for real frequencies by the Kramer-Kronig dispersion relation,

$$\text{Re } n(\omega) = 1 + \frac{2}{\pi} \text{P} \int_0^{\infty} \frac{\omega'}{\omega'^2 - \omega^2} \text{Im } n(\omega') d\omega'.$$

Write the other dispersion relation, expressing the imaginary part over the real.

- B.) Show by direct calculation with the dispersion relation that in a frequency range where resonant absorption occurs there is necessarily anomalous dispersion.
- C.) The elementary classical model for an index of refraction is based on a collection of damped electronic oscillators, described, for the k th oscillator, by

$$m \frac{d^2 \vec{x}_k}{dt^2} + m \gamma_k \frac{d \vec{x}_k}{dt} + \omega_k^2 m \vec{x}_k = e |\vec{E}_0| e^{-i\omega t}$$

where \vec{x}_k is the position vector of the k th electron with intrinsic frequency ω_k . The right hand side represents the driving force due to the incident radiation.

By employing the relations among \mathcal{D} , \mathcal{E} and polarization, and the relation between the "dynamic" refractive index n and the permittivity, show that

$$n(\omega) \approx 1 + \frac{2\pi N e^2}{m} \sum_k \frac{f_k}{\omega_k^2 - \omega^2 - i\gamma_k \omega}$$

where f_k is the number of the k th type oscillator in an atom, and N the number of atoms per unit volume. Verify that this index of refraction has the appropriate properties to satisfy the dispersion relation of (A.).

Part IV

II. Consider a grounded conducting box whose walls are at $x = 0$, $x = a$, $y = 0$, $y = b$, $z = 0$, and $z = c$. A point charge q is placed inside the box at x_0 , y_0 , z_0 .

A.) Show that the potential inside the box is given by

$$V = \frac{4q}{\epsilon_0 ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin h A_{mn} (c-z_0) \sin h A_{mn} z \cdot \sin \frac{n\pi x_0}{a}}{A_{mn} \sin h A_{mn} c} *$$

$$* \sin \frac{n\pi x}{a} \sin \frac{m\pi y_0}{b} \sin \frac{m\pi y}{b}$$

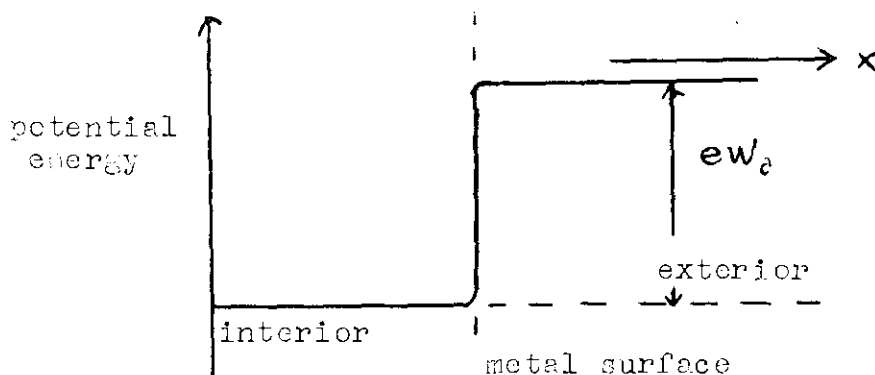
If $z < z_0$. If $z > z_0$ interchange z and z_0 .

where

$$A_{mn} = \frac{(m^2 a^2 + n^2 b^2)^{1/2}}{ab} \pi$$

B.) Now consider a conducting box bounded by 5 surfaces $x = a$, $x = z$, $y = a$, $y = 0$, and $z = 0$. There is a charge q inside this box at x_0, y_0, z_0 . Combining the solution to part (a) with the principle of images, write down the potential for the interior region $z < z_0$. Indicate how many forms of solution are needed to cover the inside of the box and why.

7. Most conduction electrons in a metal are kept from leaving the metal by a sudden rise in electric potential energy, at the surface of the metal, of an amount $e w_0$, where w_0 is the electric potential difference between the inside and the outside of the metal.



- A.) Show that if the conduction electrons inside the metal are assumed to have a Maxwell distribution of velocity, there will be a thermionic emission current of electrons from the surface of a metal at temperature T that is proportional to $\sqrt{T} e^{-ew_0/kT}$. What is the velocity distribution of these electrons just outside the surface? The measured thermionic current is proportional to

$$J = T^2 e^{-e\phi/kT}, \text{ where } \phi < w_0.$$

- B.) The conduction electrons are in reality fermions. Show that for Fermi-Dirac statistics, the thermionic current at temperature T is proportional to

$$J = T^2 e^{-e\phi/kT}$$

where $\phi = w_0 - \mu \approx w_0 - \mu_0$ is called the thermionic work function of the surface.

STATISTICAL MECHANICS

- II. A simple model for a nucleus is that of a Fermi gas, held together by a harmonic oscillator potential. The allowed energies of each nucleon are

$$\epsilon_n = (n + 3/2)\hbar\omega \quad (n = 0, 1, 2, \dots).$$

The n th quantum level has $g_n = 2(n+1)(n+2)$ distinct states. Then, start with the expression for the grand potential

$$\Omega = -kT \sum_n g_n \ln \left[1 + e^{(\mu - \epsilon_n)/kT} \right]$$

- A.) Calculate the expression for average number of particles \bar{N} .
- B.) Show that this expression is a sum of terms \bar{n}_n . Sketch roughly the plot of \bar{n}_n as a function of ϵ_n , when $kT \ll \mu$.
- C.) The energy level spacing is 10 Mev at $T=0$. Its Fermi energy $\mu_F = kT_F$ turns out to equal 50 Mev. How many nucleons are there in the nucleus?
- D.) For $0 < T \ll T_F$, the internal energy is

$$U(T) \approx \bar{N} k T_F \left[\frac{3}{5} + \left(\frac{\pi}{2}\right)^2 \left(T/T_F\right)^2 + \dots \right].$$

The nucleus of part (C) absorbs 50 Mev from an incident proton. What is its resultant temperature (give kT in Mev)?

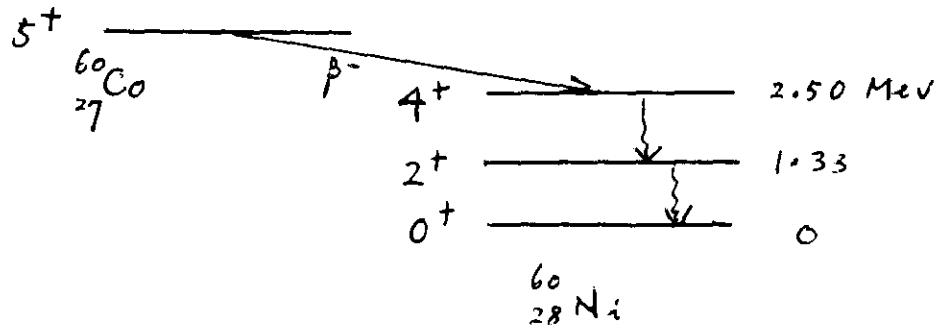
- E.) What energy must an incident proton give up to the nucleus to transform it into a Maxwell-Boltzmann gas?

- I. A.) The angular distribution of the reaction ${}^7\text{Li} (p, \alpha) {}^4\text{He}$ can be fitted with only three terms i.e.

$$1 + A(E) \cos^2 \theta + B(E) \cos^4 \theta$$

where coefficients A and B are functions of energy. What is the possible spin and parity of the level in the compound nucleus? (ground state of ${}^7\text{Li}$ is $3/2^-$)

- B.) What kind of angular distribution of gamma-rays do you expect from the decay of ${}^{60}\text{Co}$? Explain.



- II. In the lowest configuration of the nucleus ${}^{18}\text{O}$, what states can exist? (Use shell model ordering of nuclear states). How about ${}^{18}\text{F}$? In the latter case, write the total wave function of one of the states with $1 \leq j \leq 3$ (radial wave function $f(r)$ unspecified). Write your answer in terms of the spherical harmonics $Y_{lm}(\theta, \phi)$, the spin wave functions $\chi_{\text{up}}(i)$ or $\chi_{\text{down}}(i)$, and the Clebsch-Gordan coefficients $C(j_1, j_2, j_3; m_1, m_2, m_3)$.

Ph.D. Qualifying Examination

Department of Physics

Closed Book

September 6, 1967

9:00 AM

Time: 3 hours

PART A

Work 3 out of the 4 problems given. Specify which 4 you wish to be graded.

Otherwise, the first 3 only will be counted.

I. Two particles, each of mass m , interact according to the potential

$$V = c \vec{r}_1 \cdot \vec{r}_2$$

where \vec{r}_1 and \vec{r}_2 are the position vectors of the two particles and c is a positive constant.

A. Determine whether or not the quantities

$$L_{z1} = x_1 p_{y1} - y_1 p_{x1}$$

and $L_z = L_{z1} + L_{z2}$

(where the subscripts refer to the particles 1 and 2), are constants of the motion.

B. By applying the contact transformation generated by

$$F_2 = (\vec{r}_2 - \vec{r}_1) \cdot \vec{p}_1 + \frac{1}{2} (\vec{r}_1 + \vec{r}_2) \cdot \vec{p}_2$$

find the general solution for the transformed coordinates. What is the physical significance of these coordinates? (Do not transform back to the original coordinates).

II. A. (i) Use the variational principle with the trial wave function

$$\psi = \frac{1}{\pi a^3} \exp[-(r_1 + r_2)/a],$$

where a is a variational parameter, to find the energy of the ground state of helium. It is convenient to use the set of units where $\hbar = 1$, $m_e = 1/2$, and $e^2 = 2$. Ignore the exclusion principle.

(ii) In terms of the single-particle orbitals ψ_1 and ψ_2 and the spin functions α and β set up the singlet and triplet wave functions for the helium atom.

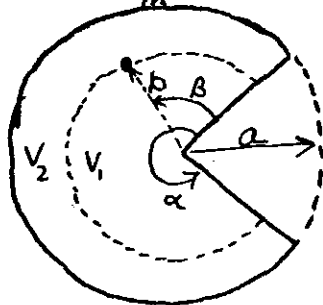
(iii) If the electrons were bosons instead of fermions, what would be the possible wave functions?

(iv) Using the Pauli spin matrices, show explicitly what the spin eigenvalues are for the singlet and triplet states. (ie., the eigenvalues of both \underline{S}^2 and S_z).

II. B. Find the electrostatic potential between two coaxial cylinders of radii $r_1 = 10$ cm and $r_2 = 2$ cm kept at potentials $U_1 = 110$ volts and $U_2 = 10$ volts, respectively.

III. Discuss in detail all physical, electromagnetic, electric, nuclear, etc effects which occur in the detection of a proton by means of a scintillation counter. In other words, the proton is first stopped in a NaI crystal. Indicate how the proton energy is transferred into light energy, how this energy is picked up by a photo-multiplier tube (discuss in detail how the photo-tube works including the polarity of the high voltage applied, etc), how the pulse goes into a pre-amplifier (give a simple circuit of the pre-amp--either tube-type or transistor-type and discuss the function of the pre-amp), discuss how the pulse is amplified and finally how it is registered on the counter. Describe how the Schmitt trigger circuit works, how one can combine these to obtain a counting circuit (in powers of 2), and indicate what modification to the circuit must be made to obtain a decade counter.

IV. Suitable circular harmonics outside a conducting wedge bounded by $\theta = 0$ and $\theta = \alpha$ are
$$V = \sum_m (A_m r^{m\pi/a} + B_m r^{-m\pi/a}) \sin\left(\frac{m\pi\theta}{\alpha}\right)$$



Using this form, find the potential (or Green's function) anywhere inside the region bounded by the grounded surfaces $\theta = 0$, $\theta = \alpha$ and $r = a$ due to a line charge at $r = b$, $\theta = \beta$, where $b < a$, and $\beta < \alpha$.

Closed Book

PART B

Work 3 out of the 4 problems given. Specify which 4 you wish to be graded. otherwise, the first 3 only will be counted.

- I. (i) The time-average potential of a neutral hydrogen atom is given by

$$\Phi(\vec{x}) = \frac{q}{r} e^{-\alpha r} \left(1 + \frac{\alpha r}{2}\right)$$

where $\alpha^{-1} = a_0/2$, and a_0 is the Bohr radius.

Show that the corresponding charge distribution is given by

$$\rho(\vec{x}) = q \delta(\vec{r}) - q \left[\frac{1}{\pi a_0^3} \right] e^{-\alpha r}$$

Discuss the significance of each term given above.

- (ii) The potential for a finite spherically symmetric charge distribution may be calculated by

$$\Phi(\vec{x}) = \int_{\infty}^{\infty} \mathbf{E} \cdot d\vec{x}$$

Show that

$$\begin{aligned} \Phi(\vec{x}) &= 4\pi \int_{\infty}^{\infty} \frac{dr''}{r''^2} \int_0^{r''} \rho(\vec{x}') r'^2 dr' \\ &= \frac{4\pi}{r} \int_0^r \rho(r') r'^2 dr' + 4\pi \int_r^{\infty} dr' r' \rho(r') \end{aligned}$$

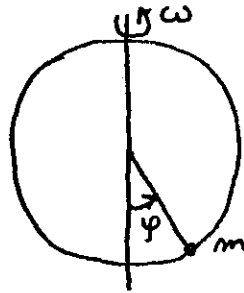
- (iii) Suppose that a sphere of radius a is uniformly charged. What is the potential at r from the center of the sphere? Plot the potential for all values of r (ie, $r > a$, $r < a$). What is the corresponding electric field?

- II. A. a. (i) Let V be a vector space of finite dimension over the complex numbers.
- (i) Define the concept of a set of generators of V .
- (ii) Define the concept of a linearly-independent set of V .
- (iii) Define the concept of a basis of V .
- (iv) Prove that if $D = (x_1, \dots, x_k)$ and $D' = (y_1, \dots, y_m)$ are bases of V , then $k = m$.
- b. Let R^3 be the usual three-dimensional coordinate space of all triplets of real numbers. Prove that the set $v_1 = (0,1,1)$, $v_2 = (1,0,1)$ and $v_3 = (1,1,0)$ form a basis for R^3 .

II. B. A particle of mass m is constrained to move on a hoop which "lies" in a vertical plane. The hoop is rotated about a vertical axis through its center at a constant angular velocity ω .

(i) Find the equation of motion for the system.

(ii) For small values of φ , what values can ω have for m to have oscillatory motion?



III. Discuss in detail the following using quantum mechanics:

- (i) Weak-field Zeeman effect
- (ii) Strong-Field Zeeman effect
- (iii) Paschen-Back Effect,
- (iv) Stark Effect
- (v) Adiabatic approximation (give application)
- (vi) Sudden approximation (give application)

IV. Calculate the eigenvalues and eigenfunctions of the operator M_x for $l = 2$.

(Express your answer in matrix form in terms of linear combinations of Y_{lm}).

(Useful relations:

$$M_x = \frac{1}{2} (L^+ + L^-)$$

$$(L^+)^j_{m+1, m} = [(j-m)(j+m+1)]^{1/2} \hbar$$

$$= 0, \text{ otherwise}$$

PART C

Work 3 out of the 4 problems given. Specify which 4 you wish to be graded. Otherwise, the first 3 only will be counted.

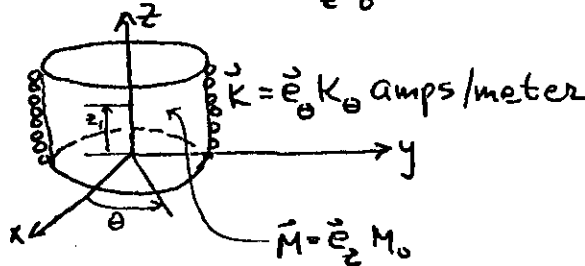
I. The normalized wave function

$$\psi = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{5}} [2e^{i3\varphi} - e^{-i\varphi}]$$

describes a state of a plane rotator of moment of inertia I . (This is a superposition of eigenstates.)

- (i) What values of angular momentum may be observed?
- (ii) What are the probabilities of finding these values?
- (iii) What values of rotational energy may be observed?
- (iv) What is the expectation value of the energy?
- (v) Give the probability distribution of the angle φ .

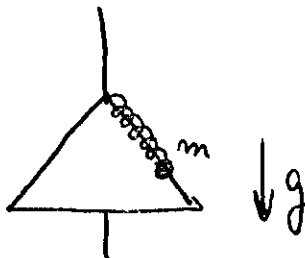
II. Suppose we have a cylinder of length l and radius a with a surface current density $\vec{K} = \hat{e}_\theta K_0$ amps/meter flowing around the cylinder. The cylinder is uniformly magnetized with magnetization $\vec{M} = \hat{e}_z M_0$. Find \vec{B} and \vec{H} along the axis at point z_1 :



Partial answer:

$$\vec{H} = \left[-\frac{2M_0}{\mu_0} + \frac{1}{2} \left(\frac{M_0}{\mu_0} + K_0 \right) \frac{(l-z_1)}{\sqrt{a^2 + (l-z_1)^2}} - \frac{1}{2} \left(\frac{M_0}{\mu_0} - K_0 \right) \frac{z_1}{\sqrt{a^2 + z_1^2}} \right] \hat{e}_z$$

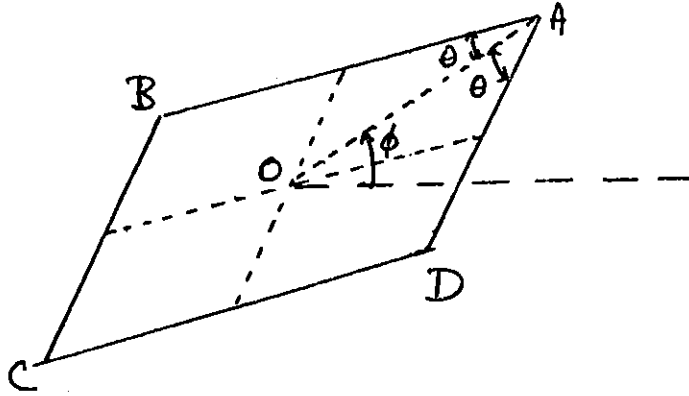
III. A.



A mass m slides on a smooth rod which forms one side of a massless equilateral triangle as shown. A spring of force constant K joins the mass to the upper vertex of the triangle. The spring has no length when unstretched.

Compute the frequency of small oscillations of the mass about a steady state when the triangle is constrained to rotate with a constant angular velocity ω about the vertical axis shown. (Use the Lagrangian formulation in setting up the equations of motion).

- III. B. Four uniform rods each of mass m and length $2a$ are pivoted together to form a parallelogram, as shown. They rotate in a horizontal plane, the center of the parallelogram being held stationary at O by 4 massless links of length a connecting O with the midpoints of the rods. If the two degrees of freedom are described by the coordinates θ, ϕ as shown, find the Lagrangian for the system.



- IV. (i) State the Ehrenfest theorem corresponding to the classical equations of motion:

$$\frac{d\vec{x}}{dt} = \frac{\vec{p}}{m} ; \quad \frac{d\vec{p}}{dt} = -\nabla V(\vec{x})$$

where $V(\vec{x})$ is the potential function.

- (ii) What is the essential condition for the validity of the classical description of the motion of a quantum particle in terms of a wave packet?
- (iii) Consider a wave packet that corresponds to a classical particle moving in a circular orbit of radius a and period T . Discuss why the classical picture of the electron moving around the proton in a hydrogen atom is untenable.

Ph.D. Qualifying Examination

Department of Physics

September 7, 1967

Closed Book

1:00 PM

PART D

Time: 3 hours

Work all 3 problems.

- I. (a) A 3-dimensional oscillator consists of a point mass m held in static equilibrium by 3 pairs of orthogonal stretched springs. The force constant of each spring in the x, y, z direction is $a, b,$ and c respectively.

Find: (i) The stress tensor \mathbb{K} relating applied force to resultant displacement $\vec{F} = \mathbb{K} \cdot \vec{R}$ using the given x, y, z axes.

(ii) The new tensor \mathbb{K}' with respect to a new coordinate system obtained by a positive rotation of 45° about the x -axis.

(iii) If $a = b$, find the relationship between the \mathbb{K}' and \mathbb{K}

- (b) Find the second-order Stark effect for a 3-dimensional isotropic harmonic oscillator in the ground state (just energy to second order).

(Hint: One-dimensional oscillator wave functions are:

$$u_n(x) = N_n H_n(\alpha x) e^{-\frac{1}{2}\alpha^2 x^2}; \quad N_n = \left(\frac{\alpha}{\pi^{1/2} 2^n n!} \right)^{1/2}; \quad \alpha = \sqrt{\frac{mK}{\hbar^2}}$$

$$E_n = (n + \frac{1}{2}) \hbar \omega \quad H_n(\xi) = (-1)^n e^{\xi^2} \frac{d^n (e^{-\xi^2})}{d\xi^n}$$

- II. (a) A heavy particle of charge ze , mass M , and nonrelativistic velocity v collides with a free electron of charge e and mass m initially at rest. With no approximations, other than that of non-relativistic motion and $M \gg m$, show that the energy transferred to the electron in this Coulomb collision, as a function of the impact parameter, is

$$\Delta E(b) = \frac{2(z e^2)^2}{m v^2} \frac{1}{b^2 + (z e^2 / m v^2)^2}$$

- (b) Discuss the difference between real forces and pseudo forces, giving examples of each type. List as many as you can of the various forces that occur in nature. What do we mean by a fundamental force, and how many of these fundamental forces occur in nature? (list them). What do we mean by the concept of a force field? Discuss these fields both classically and quantum-mechanically.

- III. The outer coating of a long cylindrical capacitor is a thin shell of radius a and the dielectric between the cylinders has an inductive capacity K on one side of a plane through the axis and K' on the other side. Show that when the inner cylinder is connected to earth and the outer has a charge q per unit length, the resultant force on the outer cylinder is:

$$\frac{q^2 (K - K')}{\pi^2 \epsilon_0 a (K + K')^2} \text{ per unit length (mks units)}$$

Thursday, January 15, 1968

(Electricity &
Magnetism)

Time: 3 hours (9:00 AM till noon)

Work all four problems.

1. The walls of a grounded rectangular conducting tube of infinite length are given by $x = 0$, $x = a$, and $y = 0$, $y = b$. A point charge is placed at $x = x_0$, $y = y_0$, and $z = z_0$ inside it. Show that the potential is given by:

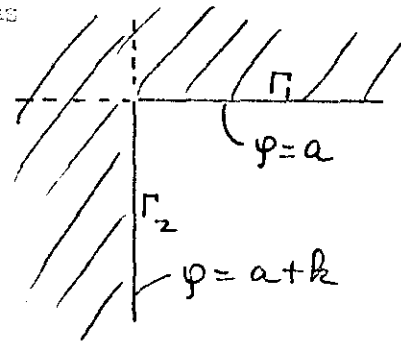
$$V = \frac{2q}{\pi \epsilon_0} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (m^2 a^2 + n^2 b^2)^{-1/2} \left\{ \exp \left[- \frac{(m^2 a^2 + n^2 b^2)^{1/2} \pi |z - z_0|}{ab} \right] \right\} \cdot \sin \frac{m\pi x_0}{a} \sin \frac{m\pi x}{a} \sin \frac{n\pi y_0}{b} \sin \frac{n\pi y}{b}$$

2. Let D be the domain $0 < \arg z < \frac{3\pi}{2}$. By employing the method of conformal mapping, find the solution of the two dimensional Laplace equation

$$\Delta \varphi = 0, \text{ (in } D \text{)}$$

subject to the boundary conditions $\varphi = a$ on Γ_1 and $\varphi = a + k$ on Γ_2 , where a and k are real constants. Draw rough diagrams

for the electrostatic potential φ and the corresponding electric field.

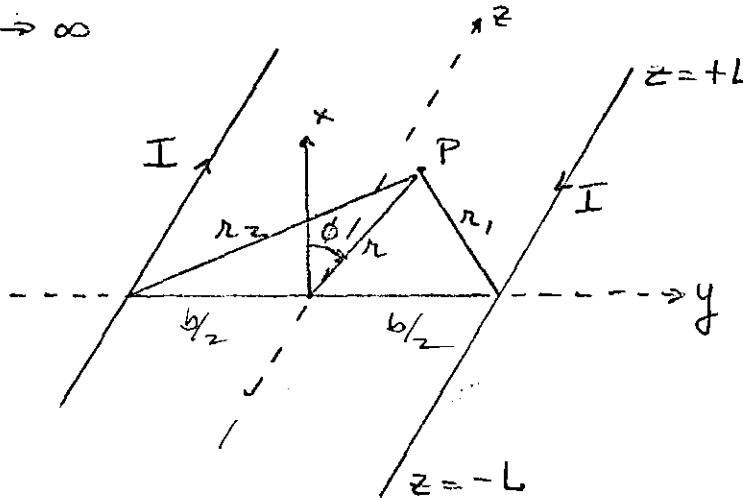


3. Two classical electrons (i.e., with no intrinsic magnetic moments) traverse a circle at opposite ends of a diameter. Compute the (instantaneous) total power radiated. Is there a dipole radiation? If not, why not? What is the nature of the lowest-order radiation in the long wave-length approximation?

4. (a) For the given figure find the vector potential at the point P,
 where the r 's are the normal distances from P to the wires.
 (b) Show that as $L \rightarrow \infty$ the vector potential becomes

$$A_z = \frac{\mu_0}{2\pi} \log \frac{r_2}{r_1}$$

- (c) Using the coordinates shown find the components of \underline{H} for the case
 $L \rightarrow \infty$



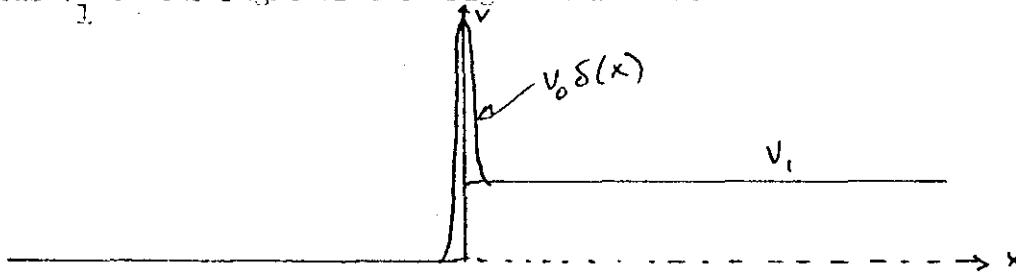
Thursday January 15, 1968

(Quantum
Mechanics)

Time: 4 hours (1:00 PM till 5:00 PM)

Part I. Work both problems

1. Suppose we have a delta-function type of potential barrier, superimposed on another potential barrier which is zero to the left of the origin and has a value V_1 to the right of the origin as follows:



The delta-function part of the potential is $V(x) = V_0 \delta(x)$ where V_0 is a constant. Calculate the reflection coefficient and transmission coefficient for this barrier, assuming that a particle of mass m is incident from left to right on this barrier. (Let $E > V_0$).

2. (a) What is the difference between a Schrödinger and Heisenberg representation of operators? (b) Show that in a Heisenberg representation, the operator

$$Q \equiv P \sin \omega t - m\omega X \cos \omega t$$

for a simple-harmonic oscillator is time-independent. (c) Is it a constant of the motion? (d) Can it be simultaneously diagonalized with the Hamiltonian?

Part II. Work two (only two) of the following four problems.

1. Consider the scattering of spinless particles by a spherically symmetric central potential V , described by the Schrödinger equation:

$$(\nabla^2 + k^2) \psi_{\vec{k}}(\vec{x}) = U(r) \psi_{\vec{k}}(\vec{x}) \quad (1)$$

$$\text{where } E = \frac{\hbar^2 k^2}{2m}, \quad U(r) = \frac{2m}{\hbar^2} V(r) \quad (2)$$

(a) Show that an integral equation equivalent to (1) is given by:

$$\psi_{\vec{k}}(\vec{x}) = \phi_{\vec{k}}(\vec{x}) + \int G_{\vec{k}}(\vec{x}; \vec{x}') U(r') \psi_{\vec{k}}(\vec{x}') d\vec{x}' \quad (3)$$

where $\phi(\vec{x})$ satisfies the homogeneous equation

$$(\nabla^2 + k^2) \phi = 0 \quad (4)$$

and $G_{\vec{k}}(\vec{x}; \vec{x}')$ is the Green's function, satisfying

$$(\nabla^2 + k^2) G_{\vec{k}}(\vec{x}; \vec{x}') = \delta(\vec{x} - \vec{x}') \quad (5)$$

(b) What is the usual asymptotic boundary condition on $\psi(\vec{x})$?

Show that the corresponding Green's function is

$$G_{\vec{k}}(\vec{x}; \vec{x}') = -\frac{1}{4\pi} \frac{e^{ik|\vec{x} - \vec{x}'|}}{|\vec{x} - \vec{x}'|} \quad (6a)$$

(c) Show that the scattering amplitude is given by

$$f(\vec{k}', \vec{k}) = -4\pi \frac{m}{\hbar^2} (\phi_{\vec{k}'}; V \psi_{\vec{k}}) \quad (7)$$

where $|\vec{k}'| = |\vec{k}|$ for elastic scattering.

Discuss the Born approximation to $f(\vec{k}', \vec{k})$ and the wave function $\psi_{\vec{k}}(\vec{x})$.

2.(A) Consider a two-dimensional isotropic harmonic oscillator,

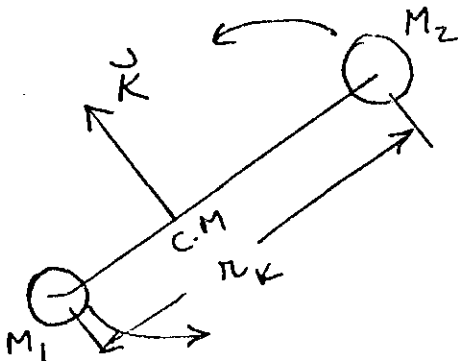
$$V = \frac{1}{2}kx^2 + \frac{1}{2}ky^2$$

- What are the energies of the lowest three levels?
- What is the degeneracy of each level?
- List all the eigenfunctions that belong to each of the three lowest levels in terms of the one-dimensional oscillator eigenfunctions designated by $\psi_n(x)$ and $\psi_m(y)$.

(B) A one-dimensional harmonic oscillator with force constant k is perturbed by a constant force. Find the perturbed wave function to first order and the perturbed ground-state energy to the second order.

Hint: $\int \psi_n(x) x \psi_m(x) dx = \left(\frac{\hbar}{2m\omega}\right)^{1/2} [\sqrt{m} \delta_{m, n+1} + \sqrt{m} \delta_{m, n-1}]$

3.



Consider a perfectly rigid rotator consisting of a "dumbbell" of length r_K having masses M_1 and M_2 attached to the two ends of a massless rod. Let us suppose that the center of mass is fixed, and that the rotation of the "dumbbell" about its own center of mass results in angular momentum \vec{K} as indicated.

(a) Show that the classical expression for the energy of rotation is

$$E_{\text{kinetic}} = \frac{|\vec{K}|^2}{2 m_p r_K^2}, \quad m_p = \frac{M_1 M_2}{M_1 + M_2}$$

(b) The quantum-mechanical treatment of the problem leads to three equations.

State very briefly in your own words where each of these equations comes from.

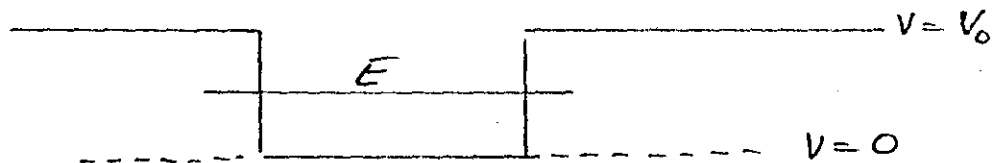
$$(i) -\frac{\hbar^2}{2m_p r_K^2} \left[\frac{\partial}{\partial r_K} \left(r_K^2 \frac{\partial}{\partial r_K} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \psi_K + V(r_K) \psi_K = E_K \psi_K$$

$$(ii) \frac{1}{r_K^2} \frac{d}{dr_K} \left(r_K^2 \frac{dR}{dr_K} \right) + \frac{2m_p}{\hbar^2} \left[E_K - U_K(r_K) - \frac{\hbar^2 K(K+1)}{2m_p r_K^2} \right] R = 0$$

$$(iii) E_K = \frac{\hbar^2 K(K+1)}{2m_p r_K^2}$$

How does this equation compare with the one obtained in part (a)?

4. Consider a particle in a one-dimensional potential well with fixed energy $E < V_0$.



- (a) Find the eigenfunctions for $E < V_0$ and sketch the first two.
- (b) Consider the limiting case where $V_0 \rightarrow \infty$. Find the corresponding eigenfunctions and eigenvalues. Again sketch the first two eigenfunctions.
- (c) How would the solutions given in (a) be changed if $E > V_0$? Give the general form of the solution and sketch one of the eigenfunctions.

How would the eigenvalues differ from those given in (a) and (b)?

Physics Ph.D. Qualifying Examination

Part I

Tuesday October 15, 1968

Time: 3 hours (9:00 A.M. till noon)

Instructions: Work all problems 1 through 5 and omit either 6 or 7.
Answer each question on a separate page.

1. (45 points) Special question for Jim McCoy.

(a) Demonstrate the mathematical transform required to go from an experimental x-ray scatter curve to a radial distribution function for all identical atoms. (b) Explain briefly the approximations required to apply this method to materials consisting of two kinds of atoms.

2. (20 points) Special question for Jim McCoy.

Give an expression for the crystal structure factor, F , defining all terms and apply to the case of a face-centered cubic lattice.

Instructions: Work problems 1 through 5 and omit either 6 or 7.
Answer each question on a separate page.

1. (45 points) Special question for Jerry Norrell.

A powdered sample of molecular weight 300 and density 2.00 exhibited the following interplanar spacings in angstrom units: 5.78, 5.01, 3.54, 3.02, 2.89, 2.50, 2.24, 2.04, 1.93, 1.77, 1.69, 1.67, 1.58, and 1.51. Determine (a) the crystal system (b) type of such system, (c) the lattice constant(s), and (d) the calculated density.

2. (20 points) Special question for Jerry Norrell.

Give an expression for the crystal structure factor, F , defining all terms, and apply to the case of a face-centered cubic lattice.

Instructions: Work problems 1 through 5 and omit either 6 or 7.
Answer each question on a separate page.

1. (45 points) Special question for George Saxon.

- (a) Derive the Hartree-Fock equations for the Helium atom.
- (b) Why will the Hartree-Fock approximation not give exact results?
- (c) Discuss, briefly, how the Hartree-Fock equations can be solved numerically.

2. (20 points) Special question for George Saxon.

Show that, if $f(x)$ is continuous, the solution of

$$g(x) = f(x) + \lambda \int_0^{\infty} \cos(2xy) g(y) dy$$

is

$$g(x) = \frac{f(x) + \lambda \int_0^{\infty} f(y) \cos(2xy) dy}{1 - \lambda^2 \pi/4}$$

assuming the legitimacy of a certain change of order of integration.

3. (5 points)

Discuss why a uniformly moving charged particle cannot radiate.

4. (60 points)

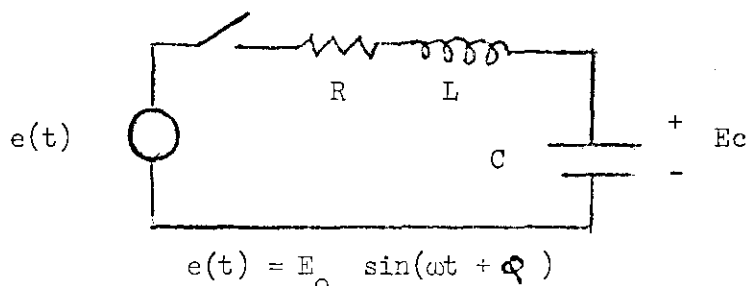
- (a) Discuss why TEM waves cannot be transmitted through a hollow perfectly conducting tube.
- (b) Consider a coaxial transmission line consisting of two perfectly conducting cylinders with radii a , b ($b > a$). Show that the time-averaged power flow of TEM wave is given by

$$P = \sqrt{\frac{\mu}{\epsilon}} \frac{a^2 c}{4} |H_0|^2 \log\left(\frac{b}{a}\right),$$

where H_0 is the peak value of the magnetic field at the surface of the inner conductor.

5. (15 points)

Consider the A-C circuit shown below. The capacitor is charged to a voltage of E_c and the switch is thrown at $t = 0$. Determine E_0 and ϕ such that there will be no transient current when the switch is thrown.



6. (20 points)

The maximum intensity per unit frequency interval ~~is~~ⁿ one sun's spectrum, occurs at wavelength of 5000 \AA . What is the surface temperature of the sun?

7. (20 points)

Suppose that one has a free Bose gas in a cubic box of volume V . The condensed particles, i.e. those in the lowest momentum state, will exert a small pressure on the walls of the box due to their zero-point motion. Show that this pressure is

$$P_0 = \frac{\pi^2 V^2}{m} \frac{N_0}{V^{5/3}}$$

where m is the mass of the particles and N_0 is the number of particles in the condensate.

Physics Ph.D. Qualifying Examinations

Part II

Tuesday October 15, 1968

Time: 3 hours (2:00 P.M. till 5:00 P.M.)

Instructions: Work all problems 1 through 4 and 2 of the 3 problems 5 through 7.

1. (45 points)
 - (a) What is the Poynting theorem?
 - (b) Illustrate its application with two examples.
 - (c) Derive it from the field equations.

2. (15 points)

Consider a sequence of measurements in connection with a spin-1/2 particle. First of all, S_z is measured and the result is $+\hbar/2$.

 - (a) What is the probability that a measurement of S_x will yield $+\hbar/2$?
 - (b) Following the S_z measurement, the particle is subjected to a magnetic field oriented along the Θ, Φ direction. This field is removed and then the S_x measurement is made. What is the probability that the result will be $+\hbar/2$?

3. (15 points)

Describe in detail why the deuteron ground-state angular wave function should be a mixture of $0.96 \ ^3S_1$ and $0.04 \ ^3D_1$ and its parity should be even.

(Useful relations: $\mu_d = +0.85735 \mu_n$, $\mu_n =$ nuclear magneton, $\mu_p = +2.79275 \mu_n$, $\mu_{neut} = -1.91315 \mu_n$)

4. (15 points)

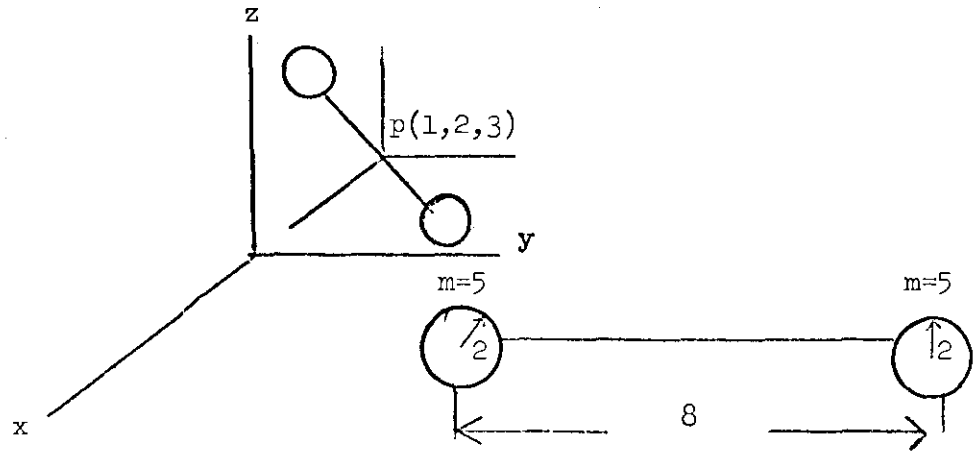
Discuss in detail the spin-orbit interaction in the hydrogen atom. Carefully indicate where all forces arise, and show quantitatively that the order of magnitude of this force is $\frac{1}{137}$ times that of the Coulomb force.

5. (45 points)

A particle of charge $+Ze$ approaches a stationary particle of infinite mass and of charge $+Z'e$. The impact parameter and the scattering angle are b and ϕ respectively. Find the distance of closest approach.

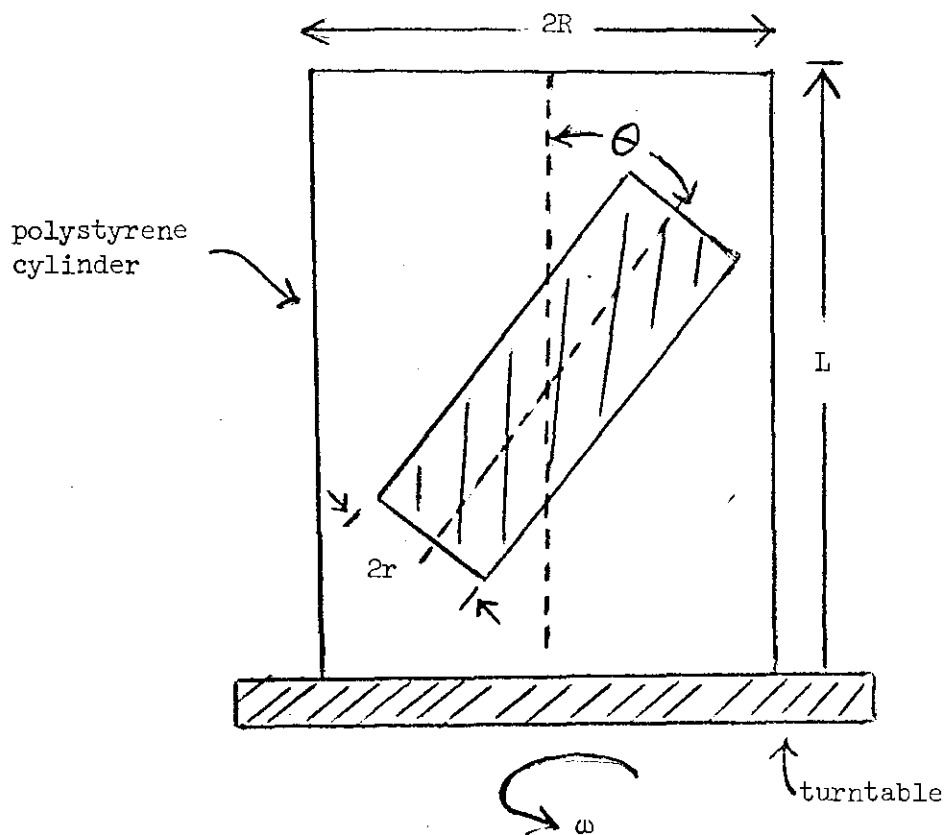
6. (45 points)

A dumbbell as shown is situated with its center at the point $P(x,y,z) = P(1,2,3)$ with its axis \parallel to the line $x = 0, y + z = 0$. Compute the tensor of inertia with respect to the x,y,z system of axes.



7. (45 points)

A cylinder of mass m , radius r , and length l is embedded in a polystyrene (massless for practical purposes) cylinder of radius R and length L . The geometrical centers of the two cylinders coincide and their axes are at an angle Θ . The system is placed on a turntable as shown. At what angular frequency, ω , will the system begin to tip?



Physics Ph.D. Qualifying Examination

Part III

Wednesday October 16, 1968

Time: 3 hours (9:00 A.M. till noon)

Instructions: Work all problems 1 through 6 and two of the three problems 7 through 9.

1. (10 points)

- (a) Why is the sky blue?
(b) What causes a mirage?

2. (15 points)

(15 min.) Give 3 methods of determining the size (or radius) of a nucleus.

3. (20 points)

Let W and W_0 be respectively the total and rest energy of a two particle system. Show that if $W \gg W_0$, the maximum energy in the zero-momentum frame is $\sqrt{2}W_0$.

4. (20 points)

Consider a system of N independent particles. Each particle can have only one of two energy levels, $-\epsilon_0$ or $+\epsilon_0$. Find the thermodynamic weight, Ω , and the entropy, S of a state with total energy $E = M\epsilon_0$.

Derive the thermodynamic properties of this system for the range $E \ll 0$. In particular, derive the relation between energy and temperature and derive an expression for the heat capacity. Observe that the heat capacity has a peak at some finite value of the temperature. This interesting behavior is called a Schottky anomaly and it is observed in some solids at very low temperature.

5. (30 points)

By employing an appropriate quantum statistics show that the thermionic emission current from the metal surface at temperature T is proportional to $T^2 e^{-\phi/kT}$, where ϕ is the so-called thermionic work function of the surface.

6. (20 points)

The number of random walks in one dimension with return to the origin in n steps and with step lengths of 0, 1, and 2 allowed is

$$N(n) = \frac{1}{2\pi} \int_0^{2\pi} (e^{2i\theta} + e^{i\theta} + 1 + e^{-i\theta} + e^{-2i\theta})^n d\theta.$$

Using the method of steepest descents, show that for large n

$$N(n) \sim 5^n (4\pi n)^{-1/2}.$$

7. (35 points)

A particle of mass m moves in an infinite potential well

$$V(x) = \begin{cases} +\infty & x < 0 \\ 0 & 0 < x < L \\ +\infty & x > L \end{cases}$$

Now the well is modified to become

$$V(x) = \begin{cases} +\infty & x < 0 \\ V & 0 < x < L/2 \\ 0 & L/2 < x < L \\ +\infty & x > L \end{cases}$$

Compute the ground-state energy for the modified well to second order in V . You may solve this problem in either one of two ways. The desired result can be obtained by a straightforward application of Rayleigh-Schrodinger perturbation theory. Another way would be to set up the transcendental equation whose solution gives the exact energy levels of the system and then to develop the power series expansion from that transcendental equation. The following sum may be useful.

$$\sum_{n=1}^{\infty} \frac{1}{(4n^2 - 1)^3} = \frac{32 - 3\pi^2}{64}$$

(Jolly, Summation of Series, p. 70)

8. (35 points)

A pendulum of mass M , charge q_0 , and length L hangs in a uniform gravitational field. An electric field is produced which is parallel to the gravitational field. If the electric field is given by

$$E = \begin{cases} 0 & t < 0 \\ E_0 e^{-t/T} & t > 0 \end{cases}$$

calculate to second order in E_0 the probability that the pendulum initially in its ground state will remain in the ground state as $t \rightarrow \infty$. What is the second order probability of excitation? What states are excited? Is the perturbation unitary (norm preserving) to this approximation?

9. (35 points)

Consider a particle of mass m incident from the left upon a potential barrier of the form

$$V(x) = q \{ \delta(x + a) + \delta(x - a) \}$$

(q is a constant)

Determine the reflection coefficient from this barrier.

Physics Ph.D. Qualifying Examination

Part IV

Wednesday October 16, 1968

Time: 3 hours (2:00 P.M. till 5:00 P.M.)

Instructions: Work problems 1 and 2 and omit either 3 or 4.

1. (60 points)

(A) A classical particle, with charge q and mass M moves so that

$$\vec{x}(t) = \hat{e}_z z_0 \cos(\omega_0 t),$$

where

$$z_0 \ll c/\omega_0.$$

Calculate the instantaneous total power (p) radiated, by employing the following formula for the time-averaged Poynting vector

$$\bar{S} = \frac{\hat{r} k}{2\pi r^2 c} \left| \int d\vec{x}' J_{\perp}(\vec{x}') e^{-i\vec{k} \cdot \vec{x}'} \right|^2$$

where J_{\perp} is the component of the current perpendicular to \hat{r} .

(B) Consider the equivalent one-dimensional quantum mechanical harmonic oscillator problem. Discuss briefly that the transition probability for the dipole radiation is given by

$$\frac{1}{T} = \frac{4}{3} \frac{\omega^3}{\hbar c} |M_{fi}|^2$$

where M_{fi} is the relevant matrix element between $|i\rangle$ and $|f\rangle$. Calculate the total power radiated for this case. Discuss how this result compares with the classical result of (A), based on the Correspondence principle.

Note: For a quantum mechanical oscillator

$$X_{nm} = \frac{1}{\alpha} \left(\frac{n+1}{2} \right)^{\frac{1}{2}}; \quad m = n+1$$

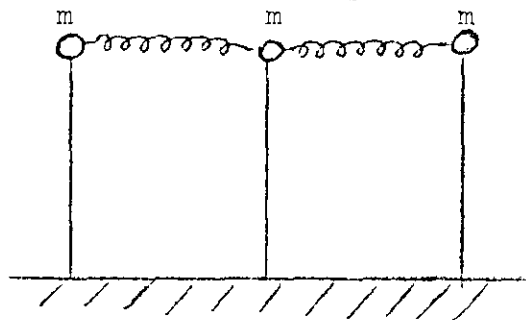
$$= \frac{1}{\alpha} \left(\frac{n}{2} \right)^{\frac{1}{2}}; \quad m = n-1$$

$$= 0 \quad \text{otherwise}$$

$$\text{where } \alpha^2 = \frac{M}{\hbar} \omega_0 c$$

2. (50 points)

Three masses are placed on the top of three equally spaced identical flexible sticks as shown. The stick exerts a force proportional to the lateral displacement. The masses are connected with identical springs. Find the motion of small vibrations. Neglect the masses of the springs and the sticks.



3. (60 points)

An infinite circular conducting cylindrical shell of radius a is divided longitudinally into quarters. One quarter is charged to a potential $+V_1$, and the one diagonally opposite to $-V_1$. The other 2 are grounded. Show that the potential of any point inside is

$$\frac{V_1}{\pi} \left(\tan^{-1} \frac{2ay}{a^2 - r^2} + \tan^{-1} \frac{2ax}{a^2 - r^2} \right)$$

4. (60 points)

A) By employing the method of images find the Green's function for the two-dimensional potential problem with the potential $V(b, \theta)$ specified on the surface of a cylinder of radius b .

B) Using the result of (A) show that the potential inside the cylinder ($r < b$) is given by Poisson integral

$$V(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} V(b, \theta_0) \frac{b^2 - r^2}{b^2 + r^2 - 2br \cos(\theta_0 - \theta)} d\theta_0$$

C) Two halves of a long conducting cylinder of radius b are separated by a small gap, and kept at different potential V_1 and V_2 . Show that the potential inside is given by

$$V(r, \theta) = V_1 + V_2 + \frac{V_1 - V_2}{\pi} \tan^{-1} \left[\frac{2br \cos \theta}{b^2 - r^2} \right]$$

D) Find the surface charge density on each half of the cylinder.

Note:

$$\int \frac{d\phi}{A + B \cos \phi} = \frac{2}{\sqrt{A^2 - B^2}} \tan^{-1} \frac{(A-B) \tan \phi/2}{\sqrt{A^2 - B^2}}$$

Time: 3 Hours

QUANTUM MECHANICS
Ph.D. Qualifying Examination

Open Schiff
Open Messiah
Open Leighton

Note: The time anticipated for working each problem is given in parenthesis. It is suggested that you spend no more time per problem than is allocated.

1. (20 minutes) A certain fine-structure multiplet of levels corresponds to $L = 3$, $S = 3/2$, and $I = 3/2$. (a) Find the number of components and give their spectroscopic designations for the fine-structure multiplet. (b) Determine into how many of these levels will further be split by hyperfine-structure splitting. (c) What is the total degeneracy of each fine-structure level?

2. (20 minutes) A one-dimensional anharmonic oscillator has a classical equation of the form:

$$m \ddot{x} + kx + ax^5 = 0.$$

Determine the energy of the 3rd excited state for this oscillator.

3. (20 minutes) Calculate in the Born Approximation the effective scattering cross section for a delta-function potential $V(\vec{r}) = a \delta(\vec{r})$.

4. (20 minutes) Consider a particle whose Hamiltonian is $H = (p^2)/(2m) + V(\vec{r})$.

Determine whether (a) the kinetic energy, (b) the total energy, (c) the parity are constants of the motion for:

- (i) $V = kx^2$; (ii) $V = k \exp(-x^2)$; (iii) The field V of an infinite plane;
(iv) The field V of an infinite half-plane.

5. (20 minutes) Find the differential cross section for a real potential $V(r) = -V_0 \exp(-r/a)$, using the Born approximation. What is the validity criterion for this case, and under what circumstances is it satisfied?

6. (5 minutes) At what energy range (high or low) is the method of partial waves useful? Why is it particularly suitable for the analysis of nuclear collisions?

7. (30 minutes) The Hamiltonian $H = (p^2)/(2m) + V(\vec{x})$ has a set of eigenkets with eigenvalues E_n . Show that, if $|m\rangle$ is any ket that has a discrete eigenvalue,

$$\sum_n (E_n - E_m) |\langle m | x | n \rangle|^2 = \frac{\hbar^2}{2m}$$

where x is a Cartesian component of \vec{x} .

8. (45 minutes) Determine the kernel (or Green's function) for a Harmonic oscillator whose Lagrangian is:

$$L = \frac{m}{2} \dot{x}^2 - \frac{m\omega^2}{2} x^2$$

Express your answer in terms of $K(x_2, x_1; t_2, t_1)$ and $T = t_2 - t_1$.

HINT: Feel free to work the problem using the classical action S if you covered Green's functions in terms of S in your classical mechanics. If you did not use Green's functions there, you may find the following relation helpful (from Morse and Feshbach, p. 786):

$$\frac{e^{-(x^2+y^2+2xyz)/(1+z^2)}}{\sqrt{1+z^2}} = e^{-x^2-y^2} \sum_{m=0}^{\infty} \left(\frac{z^m}{2^m m!} \right) H_m(x) H_m(y).$$

Ans.: I get:

$$K(x_2, x_1; t_2, t_1) = \left(\frac{m\omega}{2\pi\hbar \cos\omega T} \right)^{1/2} \exp \left\{ \frac{m\omega [(x_2^2 + x_1^2) i \sin\omega T - 2x_2 x_1]}{2\hbar \cos\omega T} \right\}$$

$$T \equiv t_2 - t_1.$$

Time: 3 hours

Ph.D. Qualifying Examination

Open Jackson

Note: The time anticipated for working each problem is given in parenthesis.

Work ALL problems.

1. (Each part is 5 to 10 minutes).

- (a) If a particle with charge q is moving with velocity \vec{v} in the presence of a static \vec{E} -field, what is the rate at which the \vec{E} -field does work?
- (b) What kind of resonant cavity is employed in a linear accelerator and what is the type of one mode? What measures must be taken for the second half cycle of the oscillating \vec{E} field when its sign changes, thus decreasing the acceleration gained by the particle in the first half cycle?

2. (less than 30 minutes) Consider an infinitely long straight wire of circular cross section of radius a with a circular hole of radius b drilled parallel to the cylindrical axis. The wire carries a current with uniform density \vec{J} .

- (a) Suppose the hole is concentric to the wire. What are the values of $\vec{B}(\vec{x})$ in the three regions $r > a$, $a > r > b$, $b > r$?
- (b) Now suppose that the radius of the hole, b , is allowed to approach a , keeping the total current I fixed. Show that the tangential component of $\vec{B}(\vec{x})$ suffers a discontinuity across the sheath, given by

$$\vec{B}(\vec{x}) = 0, \quad r = a - \epsilon$$

$$\vec{B}(\vec{x}) = \frac{2I}{cR}, \quad r = a + \epsilon$$

3. (40 minutes) Two particles with charges q_1 and q_2 and masses m_1 and m_2 collide under the action of electromagnetic forces. Consider the angular and frequency distributions of the radiation emitted in the collision.

- (a) Show that for non-relativistic motion the energy radiated per unit solid angle per unit frequency interval in the center-of-mass coordinate system is given by:

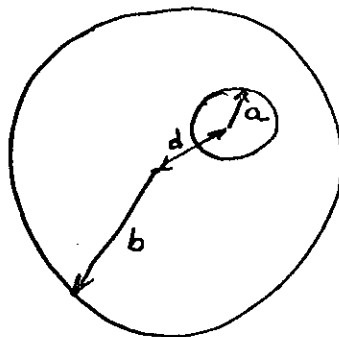
$$\frac{dI(\omega, \Omega)}{d\Omega} = \frac{\mu^2}{4\pi^2 c^3} \left| \int dt e^{-i\omega t} \ddot{\vec{x}} \times \hat{n} \left(\frac{q_1}{m_1} e^{i(\frac{\omega\mu}{c m_1}) \hat{n} \cdot \vec{x}(t)} - \frac{q_2}{m_2} e^{i(\frac{\omega\mu}{c m_2}) \hat{n} \cdot \vec{x}(t)} \right) \right|^2$$

where $\vec{x} = (\vec{x}_2 - \vec{x}_1)$ is the relative coordinate, \hat{n} is a unit vector in the direction of observation, and μ is the reduced mass.

- (b) By expanding the retardation factors show that, if the two particles have the same charge to mass ratio (eg., a deuteron and an alpha particle), the leading (dipole) term vanishes and the next-order term gives (see next page)

$$\frac{dI(\omega, \Omega)}{d\Omega} = \frac{\omega^2 \mu^4}{4\pi^2 c^5} \left(\frac{g_1}{m_1^2} + \frac{g_2}{m_2^2} \right)^2 \left| \int dt e^{-i\omega t} (\hat{m} \cdot \vec{x}) (\ddot{\vec{x}} \times \hat{m}) \right|^2$$

4. (20 minutes) (a) From the field equations, derive the Poynting theorem.
 (b) Describe the physical significance of each term in the theorem.
 (c) Describe specific physical situations in which each of the terms will vanish.
 (d) Describe specific physical situations in which each of the terms will not vanish.
- (5. (20 minutes) Find the electrostatic potential between two coaxial cylinder of radii $r_1 = 10$ cm and $r_2 = 2$ cm kept at potentials $U_1 = 110$ Volts and $U_2 = 10$ Volts, respectively.
6. An eccentric hole of radius a is bored parallel to the axis of a right circular cylinder of radius b ($b > a$). The two axes are at a distance d apart. A current of I amperes flows in the cylinder. What is the magnetic field at the center of the hole?



Time: 3 hours

CLASSICAL MECHANICS
AND
STATISTICAL MECHANICS

Open Goldstein
Open Kittel

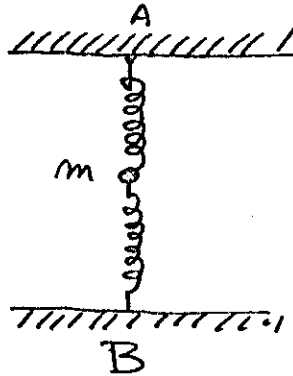
PhD Qualifying Examination

Work all four problems. The time anticipated for each problem is given in parenthesis. It is suggested that you spend no more time per problem than is allocated.

1. (50 minutes) (a) Two gravitating masses m_1 and m_2 are separated by a distance r_0 and released from rest. Find the velocities of the masses when the separation is r ($< r_0$).
- (b) A rocket moves with initial velocity v_0 toward the moon of mass M , radius r_0 . Find the cross section for striking the moon. Take the moon to be at rest and neglect all other bodies.

(40 minutes)

2. A mass m is suspended between two identical massless springs of unstretched length ℓ and spring constant k . When the mass is in equilibrium position, each spring is stretched by a distance d . Find the motion of small oscillation in a direction perpendicular to AB. Neglect gravity. Will the motion be the same if the springs are unstretched when in equilibrium?



3. (30 minutes) Two identical perfect gases with the same number of particles N and the same pressure P , but at different temperatures T_1 and T_2 are confined to vessels of volume V_1 and V_2 . The vessels are then connected.

- (a) Starting from $\Omega(E, V) = C V^N E^{3N/2} \delta E$, compute a general expression for

$$S = S(E_2, V_2) - S(E_1, V_1),$$

the change in entropy of a system in going from state 1 to state 2.

- (b) Using this expression, show that $\Delta S = 5Nk \ln \left[\frac{T_1 + T_2}{2\sqrt{T_1 T_2}} \right]$

where ΔS is the change in entropy of the combined system described above.

- (c) Prove that $\Delta S \geq 0$.

4. (one hour) Consider two spin systems A and A' placed in an external field H.

System A consists of N weakly interacting localized particles of spin 1/2 and magnetic moment μ . Similarly, system A' consists of N' weakly interacting localized particles of spin 1/2 and magnetic moment μ' . The two systems are initially isolated with respective total energies $bN\mu H$ and $b'N'\mu'H$.

They are then placed in thermal contact with each other. Suppose that $|b| \ll 1$ and $|b'| \ll 1$.

- (a) In the most probable situation corresponding to the final thermal equilibrium, how is the energy \bar{E} of system A related to the energy \bar{E}' of system A'?
- (b) What is the value of the energy \bar{E} of system A?
- (c) What is the heat Q absorbed by system A in going from the initial situation to the final situation when it is in equilibrium with A'?
- (d) What is the probability $P(E)dE$ that A has its final energy in the range between E and E + dE?
- (e) What is the dispersion $(\Delta E)^2 = \overline{(E - \bar{E})^2}$ of the energy E of system A in the final equilibrium situation?
- (f) What is the value of the relative energy spread $\Delta E / \bar{E}$ in the case when $N' \gg N$?

QUALIFYING EXAMINATION

I. (20 minutes)

- A) What is a canonical transformation?
 B) For what values of α and β do the equations

$$Q = q^\alpha \cos \beta p, \quad P = q^\alpha \sin \beta p$$

represent a canonical transformation?

[Explain clearly the procedure used to find α and β if it is not the basic definition given in part A].

II. (20 minutes)

- A) Show that the Schroedinger equation, and its complex conjugate, follow from the Lagrangian density (in one dimension)/

$$\mathcal{L} = \frac{\hbar^2}{2m} \frac{\partial \psi}{\partial x} \frac{\partial \psi^*}{\partial x} + \nabla \psi^* \psi + i \frac{\hbar}{2} (\psi \dot{\psi}^* - \dot{\psi}^* \psi)$$

- B) What are the canonical momenta?

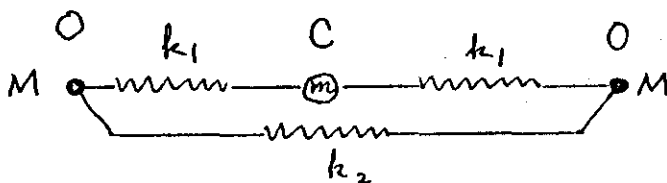
III. (20 minutes)

- A) Derive Hamilton's equations for a conservative system from Hamilton's Principle using the calculus of variations.
 B) By noticing that T is a homogeneous quadratic form in the establish the identity

$$\sum_{k=1}^n p_k \dot{q}_k = 2T$$

IV. (30 minutes)

Find the normal modes and normal frequencies for the linear vibrations of the CO_2 molecule (that is, vibrations in the line of the molecule).



QUALIFYING EXAMINATION CON'T.

V. (20 minutes)

The escape velocity of a particle on the earth is the minimum velocity required at the surface of the earth in order that the particle can escape from the earth's gravitational field. Neglecting the resistance of the atmosphere, the system is conservative. From the conservation theorem for potential plus kinetic energy show that the escape velocity for the earth, ignoring the presence of the moon, is 6.95 mi/sec.

VI. (40 minutes)

Show that if E_0 ($\gg mc^2$) is the total lab energy of electrons incident on a nucleus of mass M , the nucleus will acquire a kinetic energy of

$$T = \frac{E_0^2 (1 - \cos\theta)}{Mc^2 \left[1 + \frac{E_0}{Mc^2} (1 - \cos\theta) \right]}$$

Hint $E_0 \approx pc$, $E \approx pc$

mc, E_0, \vec{p}_0, T_i
incident electron

Nucleus

M, W, \vec{P}, T
recoil nucleus

m, E, \vec{p}, T_f
scattered electron

GENERAL QUESTIONS

VII. A) (10 minutes)

What causes an ice cream bar to "smoke"?

B) (15 minutes)

Suppose you are sitting in your car at rest waiting for a stop light to change color. Your little boy, sitting at your right hand, just releases his balloon so that it floats to the roof of your car and then comes to rest before the light changes color. When the light then changes, you immediately accelerate in the forward direction. Which way does the balloon go? Explain your answer carefully.

QUALIFYING EXAM

- I. (15 min.). Show that the principle of conservation of energy and momentum prevents pair formation from occurring in free space.
- II. (30 min.). Find the splitting of the levels of a hydrogen atom in a strong magnetic field ($\frac{e\hbar B}{2m} > |E_{nlj} - E_{n'l'j'}|$).
Hint: Neglect spin-orbit interaction energy in the first approximation, and then in the second approximation consider the spin-orbit interaction.

Also

$$\left\langle \frac{1}{r^3} \right\rangle = \frac{z^3}{a_0^3 n^3 (l + \frac{1}{2})(l+1)l}$$

$$\Delta E_S = \frac{1}{2} \frac{ze^2}{4\pi\epsilon_0 m^2 c^2} \left\langle \frac{1}{r^3} \right\rangle \langle \vec{\sigma} \cdot \vec{\ell} \rangle$$

Ans:

$$E_{nlm_l m_s} = E_n + \frac{eB\hbar}{2m} (m_l + 2m_s) + \frac{\delta E m_l m_s}{(l + \frac{1}{2})}$$

$$\text{where } \delta E = \frac{z^2 e^2 \hbar^2}{8\pi\epsilon_0 m^2 c^2 a_0^3 n^2 l(l+1)}$$

- III. (10 min.). Four independent angular momenta have quantum numbers 3/2, 6, 5/2, and 3. What quantum numbers can the resultant of these angular momenta possess? What would be the degree of degeneracy of such a system?
- IV. (40 min.). Show that the process $\pi^- + d \rightarrow n + n$ is forbidden for a scalar π^- -meson (ie., a meson which has a spin 0 and has even parity), assuming that it is captured by the S-level of "mesic" deuterium.
- V. (30 min.).
- A) Discuss concisely why the method of the partial waves is useful for the analysis of the low energy nuclear collision.

- B) Consider the neutron and alpha particle collision: By observing the elastic scattering differential cross section in the center of mass system the experimenter concluded that the scattering is not entirely S-wave type. What circumstance compels him to do so? Write down the general form of the differential scattering amplitude which includes both s- and p-waves. Suppose you want to know the contribution of the s-wave scattering only. How would you arrange the experimental setup?

VI. (40 min.).

- A) Let the two matrices A and B be related by

$$B = e^A .$$

Show that

$$\det B = e^{\text{trace } A} .$$

- B) Show that trace A is invariant under a similarity transformation,

i.e.,

$$\text{trace } A' = \text{trace } A$$

$$\text{when } A' = S A S^{-1}$$

where S is a nonsingular matrix.

- C) If A and B are two operators which both commute with their commutator; [A, B], prove the identity

$$e^A e^B = e^{A+B + 1/2 [A, B]} .$$

I (35 min.) Using the method of virtual quanta, discuss the relationship between the cross section for photodisintegration of a nucleus and electrodisintegration of a nucleus.

A) Show that, for electrons of energy $E = \gamma mc^2$, the electrodisintegration cross section is approximately given by

$$\sigma_{el}(E) = \int_{\omega_{\gamma}}^{E/\hbar} \sigma_{photo}(\omega') N(\omega') d(\hbar\omega'), \quad (1)$$

where the number spectrum for the virtual photon is approximately given by

$$N(\omega) = \frac{2}{\pi} \left(\frac{e^2}{\hbar c} \right) \frac{1}{\hbar\omega} \ln \left(\frac{a\gamma mc^2}{\hbar\omega} \right), \quad (a \approx 1), \quad (2)$$

as long as $\omega \ll \gamma mc^2/\hbar$. In Eq (1) $\omega_{\gamma}\hbar$ is the threshold energy for the photodisintegration reaction of a nucleus. Often it is possible to measure directly $\sigma_{el}(E)$ and then proceed to determine $\sigma_{photo}(E)$; in which case Eq (1) becomes an integral equation. What is the name of such an integral equation? How would you solve for $\sigma_{photo}(E)$?

B) Assuming that $\sigma_{photo}(E)$ has the resonance shape

$$\sigma_{photo}(\omega) \approx \frac{A}{2\pi} \frac{e^2}{Mc} \frac{\Gamma}{(\omega - \omega_0)^2 + (\Gamma/2)^2},$$

where $\Gamma \ll \omega_0 - \omega_{\gamma}$, sketch the behavior of $\sigma_{el}(E)$ as a function of E and show that for $E \gg \hbar\omega_0$

$$\sigma_{el}(E) \approx \frac{2A}{\pi} \left(\frac{e^2}{\hbar c} \right) \left(\frac{e^2}{Mc} \right) \frac{1}{\omega_0} \ln \left(\frac{aE^2}{mc^2 \hbar\omega_0} \right).$$

II (15 min.) Show that the wave equation

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \psi(\vec{x}, t) = 0$$

is Lorentz covariant.

III (20 min.) Show that in general a long, straight bar of uniform cross-sectional area A with uniform lengthwise magnetization M , when placed with its flat end against an infinitely permeable flat surface, adheres with a force given approximately by

$$F \simeq 2\pi A M^2$$

IV (40 min.) Using the Lienard-Wiechert (L-W) fields, discuss the time-average power radiated per unit solid angle, $\langle \frac{dP}{d\Omega} \rangle_t$, in nonrelativistic motion of a particle with charge e , moving in a circle of radius R in the x - y plane with constant angular frequency ω_0 . Sketch the angular distribution of the radiation and determine the total power radiated. Hint. Note that

$$\langle \frac{dP}{dt} \rangle_t = \frac{c}{4\pi} \frac{1}{r} \int_0^\pi dt |r E_a|^2 = \frac{c}{4\pi r} \int_0^\pi dt \left(\frac{e}{c}\right)^2 |\hat{r} \times (\hat{r} \times \dot{\beta})|^2$$

$$\beta = v/c$$

V (40 min)

A) Show that the Lagrangian for a single particle (m, q) in an electromagnetic field is

$$L = -mc^2 \sqrt{1 - \left(\frac{v}{c}\right)^2} - q\Phi + \frac{q}{c} \mathbf{A} \cdot \mathbf{v} \quad (1)$$

What is the canonical momentum \vec{p} conjugate to \vec{r} ?

Hint. To show that (1) is the correct Lagrangian

demonstrate that the resultant Lagrange equation reduces

to the relativistic equation of motion, $\vec{F} = \frac{d}{dt} \left(\frac{m\vec{v}}{\sqrt{1-(v/c)^2}} \right)$.

Simpler way to establish eq (1) is to appeal to the notion of Lorentz invariance.

B) Show that the Hamiltonian corresponding to (1) is given by

$$H = \sqrt{(c\vec{p} - q\vec{A})^2 + (mc^2)^2} + e\Phi.$$

VI (30 min)

A) A point charge q is placed at Y measured from the center of a conducting sphere of radius R which is raised to a fixed potential V . What is the electrostatic potential at X ?

B) Show that the force on q due to the sphere at fixed potential is

$$\vec{F} = \frac{q}{y^2} \left[VR - \frac{qRy^3}{(y^2 - R^2)^2} \right] \hat{y}.$$

Discuss qualitatively the salient features of \vec{F} .

- I (15 min) Show that for a system of fermions the number of states with energy range $d\epsilon$ is given by

$$d(\epsilon)d\epsilon = (2I+1) \frac{2\pi V}{h^3} (2M)^{3/2} \sqrt{\epsilon} d\epsilon$$

where V is the quantization spatial volume, and M and I are the mass and spin of a single particle.

- II (30 min)

- A) What is the density of states for photons whose frequency range is $[\nu, \nu+d\nu]$?
- B) Show that the energy per unit volume of the radiation with frequency range $[\nu, \nu+d\nu]$ is given by the Planck's formula:

$$u(\nu, T) = \frac{8\pi h}{c^3} \frac{\nu^3}{e^{\frac{h\nu}{kT}} - 1}$$

where k is the Boltzman constant with

$$kT = 1.38 (T) \times 10^{-23} \text{ joule.}$$

- C) The maximum intensity per unit frequency interval in the sun's spectrum occurs at a wave length of 5000 \AA . What is the surface temperature of the sun?

- III (30 min) Consider a gas contained in volume V at temperature T . The gas is composed of N distinguishable particles of zero rest mass. Show that the partition function is given by

$$Z = \left[\left(\frac{kT}{c} \right) \frac{8\pi V}{h^3} \right]^N$$

where c is the velocity of light.

Find the equation of state and the internal energy U of the gas. How do the pressure and U compare with those of an ordinary gas?

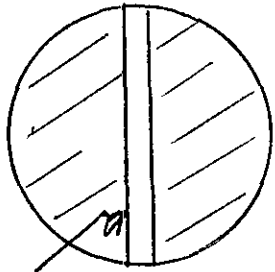
- IV (30 min) Compare the microcanonical ensemble, the canonical ensemble, and the grand canonical ensemble. What are the defining conditions for each and what are their drawbacks, if any?
- V (30 min) Each of the N particles (distinguishable particles), in a box of volume V has two possible energies, 0 or E , where E is inversely proportional to the volume ($E=b/V$, where b =constant). What is the equation of state of this system?
- VI (15 min) Why must the wave function of a system containing identical particles be either symmetric or antisymmetric?

GENERAL QUESTIONS

- A (10 min) What causes a "halo" around the moon on certain "clear" nights?
- B (15 min) (a) What causes a rainbow?
(b) What causes a double rainbow?
(c) Are the colors in the same order in a rainbow as they are in the second portion of the double rainbow?
- C (5 min) Some stars look blue against the dark of a night sky. How does the temperature of the star compare with that of the sun?

Work 5 of the 6 problems given below. Be sure to include each problem on separate paper. The time expected to work each problem is given in parenthesis.

1. (15 min) An object of mass m is dropped into a tunnel that has been dug through the center of the Earth. Assuming a uniform mass density for the Earth, show that the object will undergo simple harmonic motion. Determine its period in terms of Newton's gravitational constant, G , as well as the Earth's mass, M , and radius, R .



tunnel

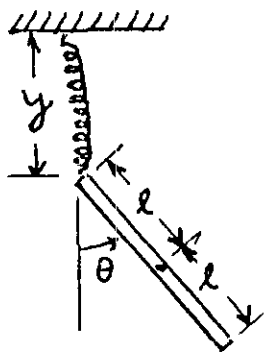
2. (20 min)

A uniform bar of mass M and length 2ℓ is suspended from one end by a spring of force constant k . The bar swings freely in one vertical plane, and the spring is constrained to move only in the vertical direction. Using the variables y and θ , the Lagrangian is:

$$L = T - V = \frac{1}{2} M \left[\frac{4}{3} \ell^2 \dot{\theta}^2 + \dot{y}^2 - 2\ell \dot{y} \dot{\theta} \sin\theta \right] - \frac{1}{2} k y^2 + (y + \ell \cos\theta) M g,$$

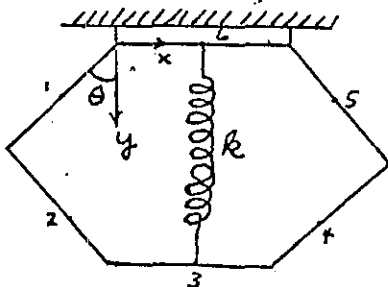
where the "dot" on the variables y and θ refers to a time derivative.

- (a) Use either the regular approach or the matrix approach to obtain the Hamiltonian. (Remember that your answer must be in terms of θ , y , p_θ , and p_y , or it is wrong!)
- (b) Use this Hamiltonian to write down Hamilton's four (4) equations of motion (do not attempt to solve the equations, just write the equations themselves).

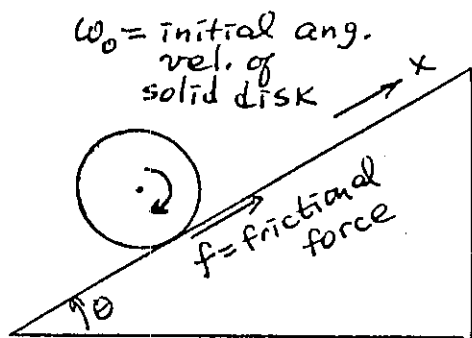


3. (15 min)

Suppose that six equal uniform rods each of length a and mass m are freely hinged at the ends to form a hexagon and hang in a vertical plane. They are supported by one side which is held horizontal and by a spring connecting the midpoints of this side to the opposite side. If the tension in the spring is k times its length, find by the principle of virtual work its length in the equilibrium position in which it is vertical and the bottom rod is horizontal.



4. (35 min)



A solid disk of mass M and radius R rotates about its horizontal axis with initial angular velocity ω_0 . The disk is rotating in the direction shown in the figure, and at $t = 0$ is placed on the incline so that it tends to roll up the plane. The coeff. of friction between the disk and the plane is μ . The disk will first travel a distance l_1 up the plane until it no longer slides. It will then roll without sliding an additional distance l_2 until it stops.

(a) Show that the angular velocity of the disk at distance l_1 is

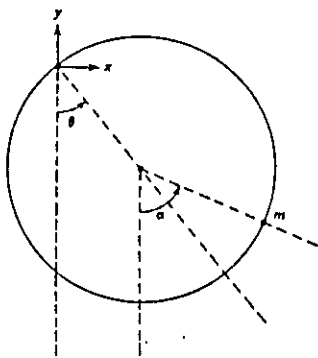
$$\omega_1 = \omega_0 \frac{[\mu \cos \theta - \sin \theta]}{[3\mu \cos \theta - \sin \theta]}$$

(b) Find the time it takes to go distance l_1 , and from v_1 and t_1 , show that l_1 is given by:

$$\frac{(\omega_1 R)^2}{2g} = \frac{(\mu \cos \theta - \sin \theta)}{(3\mu \cos \theta - \sin \theta)^2} l_1$$

(c) Find l_2 .

5. (40 min)



A bead of mass m slides without friction on a uniform circular ring of mass M and radius a . The ring oscillates (under gravity) about a fixed point on its circumference.

A. Show that the total kinetic energy of the system is given by

$$T = T_{\text{bead}} + T_{\text{ring}} = \frac{a^2}{2} \left[2M\dot{\theta}^2 + m\dot{\theta}^2 + m\dot{\alpha}^2 + 2m\dot{\theta}\dot{\alpha}\cos(\alpha - \theta) \right] \quad (1)$$

(Hint: To obtain T_{bead} first write down (x, y) coordinates of the bead in terms of (a, θ, α) . To obtain T_{ring} recall that the moment of inertia of a circular ring about a perpendicular axis through the center is given by Ma^2 .)

B. What are the values of θ and α at the equilibrium position of the bead (call it A)? Show that the potential energy relative to the point A is given by

$$V = Mga(1 - \cos \theta) + mga(2 - \cos \theta - \cos \alpha) \quad (2)$$

Draw a diagram to justify the above relation.

5. (cont'd)

C. Henceforth regard $q_1 \equiv \theta$, $q_2 \equiv \alpha$ as the generalized coordinates.

Show that near the equilibrium the Lagrangian can be put in the form:

$$L = \frac{1}{2} \sum_i \sum_j (\dot{q}_i T_{ij} \dot{q}_j - q_i V_{ij} q_j) \quad (3)$$

Find the (2x2) matrix representatives of T_{ij} and V_{ij} under the assumption of small oscillations about the equilibrium.

D. By solving the characteristic equation

$$\text{Det.} \left| V - \omega^2 T \right| = 0$$

show that the two eigenfrequencies are given by

$$\omega_1^2 = \frac{g}{2a}, \quad \omega_2^2 = \frac{m+M}{M} \frac{g}{a}$$

Find the corresponding eigenvectors $A^{(1)}$, $A^{(2)}$ and arrange them as follows:

$$\omega_1 : A^{(1)} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$$

$$\omega_2 : A^{(2)} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

$$\text{Answer: } A = (A^{(1)}, A^{(2)}) = \frac{1}{a(M+2m)^{1/2}} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\sqrt{\frac{m}{M}} \\ \frac{1}{\sqrt{2}} & \frac{m+M}{\sqrt{mM}} \end{pmatrix}$$

E. Find the normal coordinates Q_1 and Q_2 . (Simply outline the procedure without any calculation.) "Show" that

$$Q_1 = \sqrt{\frac{2}{M+2m}} a [(m+M)\theta + m\alpha]$$

$$Q_2 = \sqrt{\frac{Mm}{M+2m}} a [\alpha - \theta]$$

Give the physical meaning of Q_1 and Q_2 , by providing the geometric picture of the underlying physics.

6. (15 min) Consider the weak scattering of two particles via a potential $V(r)$ such that $V(b) \ll E$, where b is the impact parameter and E is the (positive) total energy. Show that the distance of closest approach of the two particles is given approximately by

$$r_{\min} \approx b \left(1 + \frac{V(b)}{2E} \right).$$

Work 5 of the 6 problems given below. Be sure to include each problem on separate paper. The time expected to work each problem is given in parenthesis.

1. (25 min) The unitary operator $\exp(-i\hat{H}t/\hbar)$ transforms the position and momentum operators \hat{x} and \hat{p}_x into the following Heisenberg representation:

$$\hat{x}^H = e^{i\hat{H}t/\hbar} \hat{x} e^{-i\hat{H}t/\hbar} \quad \text{and} \quad \hat{p}_x^H = e^{i\hat{H}t/\hbar} \hat{p}_x e^{-i\hat{H}t/\hbar}$$

Show that for a simple one-dimensional harmonic oscillator, and for \hat{x} and \hat{p}_x to be the usual differential operator form (in the coordinate \vec{r} -representation), the transformed operators are:

$$\hat{x}^H = \hat{x} \cos\omega t - \frac{1}{\sqrt{km}} i\hbar \sin\omega t \frac{\partial}{\partial x} \quad \text{and} \quad \hat{p}_x^H = -i\hbar \cos\omega t - \sqrt{km} \hat{x} \sin\omega t$$

2. (30 min) A particle of mass m is constrained to move along an infinite horizontal wire without friction (i.e., it is a free particle whose wave function is $u_{p_0}(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ip_0 x/\hbar}$). Suppose that a measurement on the system at $t = 0$ shows that the wave function is $\psi(x,0) = A x e^{-\alpha x^2}$. Determine the wave function as a function of time. (Hint: It is not $\psi(x,t) = A x e^{-\alpha x^2} e^{-iE_0 t/\hbar}$!! You should use either the Green's function method or some other approach and then verify that your answer agrees to the statement of the problem by letting $t \rightarrow 0$).

3. (30 min) A charged particle ($m, -e$) is in the presence of a magnetic field \vec{B} described by the Hamiltonian: $\mathcal{H} = \frac{1}{2m} \left(\hat{p} - \frac{e}{c} \vec{A} \right)^2 + V(r)$, (1)

where \vec{A} is the vector potential associated with \vec{B} .

- (A) Employ the (Heisenberg) equation of motion and show that

$$\vec{v} \equiv \frac{d\vec{r}}{dt} = \frac{1}{m} \left(\hat{p} - \frac{e}{c} \vec{A} \right). \quad (2)$$

- (B) Evaluate the commutator $[v_x, v_y] = () B_z$, etc, (3)

where you are supposed to find the quantity in the bracket.

- (C) Discuss the physical meaning of (3). It might be helpful if you refer to

$$(\Delta v_x)(\Delta v_y) \geq (\text{something}), \quad (4)$$

where v_x , for instance, denotes the root-mean-square deviation of v_x .

What quantity does (something) in Eq. (4) represent?

4. (35 min) An electron ($m, -e$) is a one-dimensional well

$$V(x) = \frac{1}{2} m \omega_0^2 x^2$$

is immersed in a constant, uniform electric field of magnitude \mathcal{E} which points in the x -direction, resulting in a perturbation

$$H' = e \mathcal{E} x$$

The total Hamiltonian is then given by

$$H = \frac{p^2}{2m} + V + H'. \quad (1)$$

The matrix elements of x relative to the energy eigenfunctions $E_n = n$ are known to be

$$\begin{aligned} \langle n | x | k \rangle &= \sqrt{\frac{n+1}{2\alpha}} \quad , \quad k = n+1 \\ &= \sqrt{\frac{n}{2\alpha}} \quad , \quad k = n-1 \end{aligned} \quad \left(\alpha = \frac{m \omega_0}{\hbar} \right) \quad (2)$$

- A. By availing yourself of the matrix elements of x given in (2) evaluate

$$\langle n | x^2 | n \rangle .$$

- B. Employ time-independent perturbation theory to

(i) find the first-order corrections to the energy,

(ii) calculate the second-order corrections to the energy.

- C. Show that the present problem can be solved exactly for the energy E_n of the n th eigenstate. Compare the exact result for E_n with that obtained in part (B). Discuss.

5. (20 min) Suppose $\psi(\vec{x}, t)$ is a time-dependent wave function describing a nonrelativistic, quantum mechanical particle of mass m in an external field, $V(\vec{x})$. Show that

$$(a) \quad \frac{d}{dt} \langle \vec{x} \rangle = \frac{1}{m} \langle \vec{p} \rangle$$

$$(b) \quad \frac{d}{dt} \langle \vec{p} \rangle = - \langle \vec{\nabla} V(\vec{x}) \rangle$$

Here $\langle \quad \rangle$ denotes an expectation value. For example,

$$\langle \vec{x} \rangle = \int \psi^*(\vec{x}, t) \vec{x} \psi(\vec{x}, t) d^3x.$$

6. (20 min) From the one-dimensional Schrödinger equation,

$$i\hbar \frac{\partial \psi(x, t)}{\partial t} = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \psi(x, t) .$$

Show that the continuity equation $\frac{\partial \rho(x, t)}{\partial t} + \frac{\partial j(x, t)}{\partial x} = 0$ holds, where

$$\rho(x, t) = \psi^*(x, t) \psi(x, t) \quad ,$$

$$j(x, t) = \frac{\hbar}{m} \operatorname{Im} \left[\psi^* \frac{\partial \psi}{\partial x} \right] .$$

Work 5 of the 6 problems given below. Be sure to include each problem on separate paper. The time anticipated to work each problem is given in parenthesis.

1. (20 min) Given $\nabla^2 \phi(\vec{x}) = 4\pi \rho(\vec{x})$ in some volume V bounded by a surface S , show that

$$\phi(\vec{x}) = \int_V d^3x' G(\vec{x}, \vec{x}') \rho(\vec{x}') - \frac{1}{4\pi} \oint_S \phi(\vec{x}') \hat{n}' \cdot \nabla' G(\vec{x}, \vec{x}')$$

given a function $G(\vec{x}, \vec{x}') = G(\vec{x}', \vec{x})$ which satisfies

$$\nabla^2 G(\vec{x}, \vec{x}') = -4\pi \delta(\vec{x} - \vec{x}')$$

in V and $G(\vec{x}, \vec{x}') \Big|_{\vec{x} \in S} = 0$ on S .

2. (35 min)

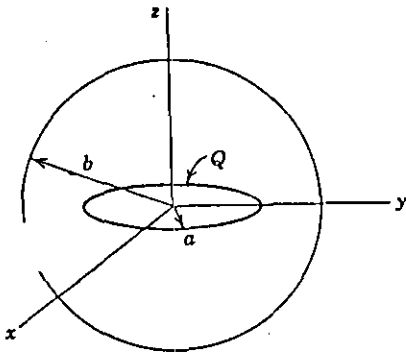
Consider a system consisting of a hollow grounded sphere of radius b with a concentric circular ring of radius a and total charge Q (See the diagram shown.) The ring of the charge is located in the x - y plane.

- A. Observe that the potential at the field point P is given by

$$\Phi(\vec{r}) = \int \frac{\rho(\vec{r}_0) d^3x_0}{|\vec{r} - \vec{r}_0|} + \sum_n (A_n r^n + B_n / r^{n+1}) P_n(\cos \theta) \quad (1)$$

where $\rho(\vec{r}_0)$ represents the charge density of the ring.

- (i) Explain why do you need the summation terms in (1)? One the two terms, either A_n or B_n must vanish identically. Why?
- (ii) Write down the expression of $\rho(\vec{r}_0)$ in spherical coordinates.
- (iii) By imposing the boundary condition for $\Phi(\vec{r})$ at $r = b$ determine the expansion coefficients in (1). Hint: You need to make use of the multipole expansion of the first term.
- B. Find the charge density σ on the sphere. Sketch the angular distribution $\sigma(b, \theta)$, and discuss its salient feature. What is the total charge on the sphere? Why?
- C. What is the electric field \vec{E} outside the sphere? Why? What is the potential outside the sphere? What sort of the role does the grounded sphere play regarding the charge inside? Find the electric field $\vec{E}(0)$ at the center of the sphere. Discuss.



3. (25 min) A sphere of radius a carries a uniform charge density, σ . The sphere is rotated about a diameter with constant angular velocity, ω , about the z-axis. Given that the vector potential is (take the origin of coordinates at the sphere's center):

$$\vec{A}(\vec{x}) = \frac{1}{c} \int da' \frac{\vec{K}(\vec{x}')}{|\vec{x} - \vec{x}'|}, \quad \text{where } \vec{K}(\vec{x}') \text{ is the surface current on the sphere,}$$

Show that at the origin,

$$\vec{B}(0) = \frac{8 \pi \sigma a \omega}{3c} \hat{k}, \quad \text{where } \hat{k} \text{ is a unit vector in the z-direction.}$$

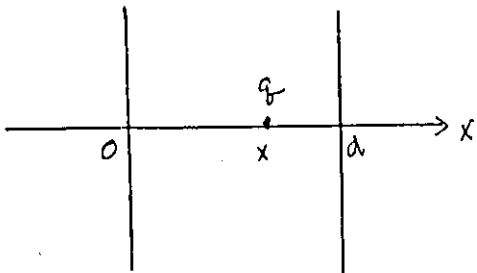
4. (25 min)

- A. Prove Green's Reciprocation theorem: If Φ is the potential due to a volume-charge density ρ within a volume V and a surface-charge density σ on the conducting surface S bounding the volume V , while Φ' is the potential due to another charge distribution ρ' and σ' , then

$$\int_V \rho \Phi' d^3x + \int_S \sigma \Phi' da = \int_V \rho' \Phi d^3x + \int_S \sigma' \Phi da \quad (1)$$

- B. Two infinite grounded parallel conducting planes are separated by a distance d .

A point charge q is placed between the planes. Use the theorem established in part (A) to prove that the total charge on one of the planes is equal to $(-q)$ times the fractional perpendicular distance of the point charge from the other plane. (Hint: Choose as your comparison electrostatic problem with the same surfaces one whose charge densities and potential are known and simple.)



page 3
Electricity and Magnetism
Part III

5. (30 minutes) This problem deals with a cylindrical capacitor. The cylinders are displaced so that there is a distance δ between the axes of the cylinders (as shown in the given figure). The inner shell has a radius ρ_1 and is kept at a potential V_1 while the outer shell has a radius ρ_2 and is kept at potential V_2 . Assume that $\delta \ll \rho_1$.

The axial case ($\delta = 0$).

- (i) Starting with the Laplace equation in cylindrical coordinates, show that the general solution for the potential between the cylinders is given by

$$\Phi(\rho) = A_0 + A'_0 \ln \rho \quad (1)$$

- (ii) Calculate the constants in Eq. (1), for the given situation, and use these to state the potential in terms of V_1 , V_2 , ρ_1 , ρ_2 and ρ .

- (iii) The term proportional to $\ln \rho$ in the solution can be shown to be due to a charge density. Show this by calculating the charge density, σ_1 , for the case when $\rho = \rho_1$.

The nonaxial case ($\delta \neq 0$).

- (iv) Argue that due to the symmetry of the problem, the potential will satisfy the equation

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} = 0 \quad (2)$$

Employ the method of separation of variables to show that the most general solution for Eq. (2) is given by

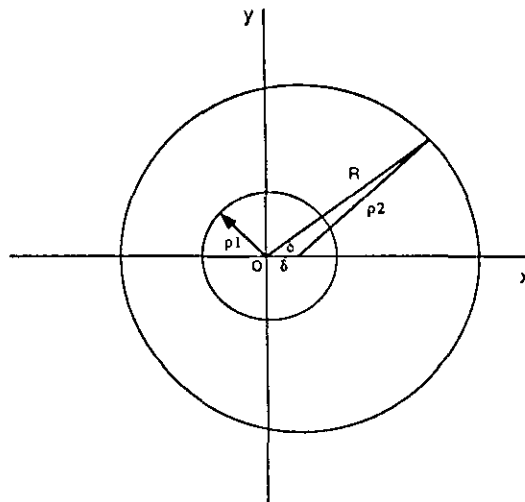
$$\Phi = \sum_{n=1}^{\infty} \left(A_n \cos(n\phi) + B_n \sin(n\phi) \right) \rho^n + \sum_{n=1}^{\infty} \left(A'_n \cos(n\phi) + B'_n \sin(n\phi) \right) \rho^{-n} + A_0 + A'_0 \ln \rho \quad (3)$$

- (v) Use Eq. (3) to obtain a general solution, for the potential, correct to a first order correction in δ .

- (vi) Now employ the boundary conditions to show that the potential can be written in the form

$$\Phi(\rho, \phi) = V_1 + \frac{V_2 - V_1}{\ln\left(\frac{\rho_2}{\rho_1}\right)} \ln\left(\frac{\rho}{\rho_1}\right) - \frac{\delta}{\rho_2^2 - \rho_1^2} \left[V_1 - \frac{V_2 - V_1}{\ln\left(\frac{\rho_2}{\rho_1}\right)} \right] \left[\rho - \frac{\rho_1^2}{\rho} \right] \cos \phi \quad (4)$$

- (vii) What does Eq. (4) reduce to for the special case $V_1 = V_2 = V$?



6. (15 minutes) This problem deals with the propagation of electromagnetic radiation in regions of charge and current free space. The wave equation of the electric field away from external sources and in the absence of conducting materials (for vacuum) is

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad (1)$$

(i) Assume that we are interested in monochromatic radiation so that

$$\mathbf{E} \rightarrow \mathbf{E} e^{-i\omega t} \quad (2)$$

Obtain the vector Helmholtz equation by using Eq. (2) in (1). Due to the vector nature of \mathbf{E} , this equation is not easily solved. A common approach to this problem is to transform the vector Helmholtz equation into a scalar equation which is then easily separated.

(ii) Use the transformation

$$\mathbf{E} = \mathbf{r} \times \nabla \psi \quad (3)$$

to obtain the scalar Helmholtz equation from the vector Helmholtz equation obtained in part (i). (Hint: Show that Eq. (3) can be written in the form

$$\mathbf{E} = -\nabla \times (r\psi)$$

and use this to obtain the desired result.) What does the scalar Helmholtz equation become, in the limit, $\omega \rightarrow 0$?

Work 5 of the 6 problems given below. Be sure to include each problem on separate paper. The time anticipated to work each problem is given in parenthesis.

1. (35 min) The canonical form of the density operator is given by

$$\hat{\rho} = A e^{-\hat{H}/kT} \equiv A e^{-\beta \hat{H}} \quad \left(\beta \equiv \frac{1}{kT} \right) \quad (1)$$

Consider that \hat{H} is the Hamiltonian of one-dimensional harmonic oscillator with the fundamental frequency ω_0 .

- A. Find the diagonal elements of $\hat{\rho}$ relative to the energy eigenvectors $|E_n\rangle \equiv |n\rangle$.
- B. Evaluate the normalization constant A by imposing the condition $\text{Tr } \hat{\rho} = 1$.
What does the last condition mean? How is A related to the quantum partition function Q? Find the explicit form of Q for a harmonic oscillator and show that

$$Q = \frac{1}{2 \sinh \beta \hbar \omega_0 / 2} \quad (2)$$

- C. Calculate the energy expectation value $\langle E \rangle$ of the oscillator.
- D. Henceforth consider a system of N noninteracting identical oscillators. Calculate the internal energy U of the system, and thence the heat capacity C_V .
- E. Calculate the Helmholtz free energy F and the entropy S.
- F. Write down the classical partition function for a system of N identical classical oscillators. Evaluate U, C_V , F, and S for them, and demonstrate that they agree with the classical limits of the quantal counter parts.

2. (10 min) Given the energy-momentum relationship for an electron in a one-dimensional lattice to be $E = p^2/2m + A p^4 + B p^6$, where A and B are constants.

- (a) Find the phase velocity v_p
- (b) Find the group velocity v_g
- (c) Find the effective mass m^*

(Express all answers in terms of p, A, and B).

3. (30 min)

Assume that there exists an infinite of energy levels equally spaced 1 eV apart starting at 0 eV. Assume further that there are 3 particles which share a total of 10 eV. Finally, assume that all possible arrangements are equally likely.

- (a) If the particles obey Fermi-Dirac statistics, show all possible arrangements in the levels.
- (b) What is the probability of finding a particle in level 0? in level 1? in level 2? in level 3? in level 4? in level 5? in level 6? in level 7? in level 8? in level 9? in level 10?
- (c) If the particles satisfy Bose-Einstein statistics (and still share a total of 10 eV), how many arrangements are there? (Show these arrangements in a diagram similar to that in part (a).)
- (d) For case (c), what is the probability of finding 2 particles in the same level?
- (e) If the particles were distinguishable, and the limitation on the total energy is removed, how many arrangements would there be if you could place the distinguishable particles in any of the levels from 0 eV to 10 eV?

4. (30 min)

A simple model for a nucleus is that of a Fermi gas, held together by a harmonic oscillator potential. The allowed energies of each nucleon are $\epsilon_n = (n+3/2)\hbar\omega$ ($n = 0, 1, 2, \dots$). The n th quantum level has $g_n = 2(n+2)(n+1)$ distinct states. Then, start with the expression for the grand potential

$$\Omega = -kT \sum_n g_n \ln \left[1 + e^{(\mu - \epsilon_n)/kT} \right]$$

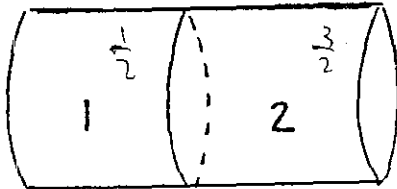
- A. Calculate the expression for the average number of particles \bar{N} .
- B. Show that this expression is a sum of terms \bar{n}_n , mean number density distribution. Sketch the plot of \bar{n}_n as a function of ϵ_n , when $kT \ll \mu$.
- C. The energy -level spacing $\hbar\omega$ for a given nucleus is 10 MeV at $T = 0$. Its Fermi energy $\mu_F = kT_F$ turns out to equal 50 MeV. How many nucleons are there in the nucleus?
- D. For $0 < T \ll T_F$, the internal energy is

$$U(T) \simeq \bar{N}kT_F \left[\frac{3}{5} + \left(\frac{\pi^2}{4}\right) \left(\frac{T}{T_F}\right)^2 + \dots \right]$$

The nucleus of part C absorbs 50 MeV from an incident proton. What is its resultant temperature? (Give kT in MeV.)

5. (30 min). A cylinder is separated into two compartments by a free sliding piston. Two ideal Fermi gases are placed into the two compartments, numbered 1 and 2. The particles in compartment 1 have spin $1/2$, while those in compartment 2 have spin $3/2$. They all have the same mass. Find the equilibrium relative density of the two gases at $T = 0$ and at $T \rightarrow \infty$.

Hint:



Recall that the internal energy of a Fermi gas at low temperatures is given by

$$U = \frac{3}{5} N \epsilon_F \left[1 + \frac{5\pi^2}{12} (kT/\epsilon_F)^2 + \dots \right] \quad (A)$$

where ϵ_F denotes the Fermi energy:

$$\epsilon_F = \frac{\hbar^2}{2m} (6\pi^2/gv)^{2/3} . \quad (B)$$

In Eq. (B) g denotes the multiplicity and v stands for the average volume occupied by a single particle.

6. (10 min)

A. Show that $\theta(x)$ defined by

$$\theta(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dk \frac{e^{ikx}}{k - i\epsilon} \quad (\epsilon > 0) \quad (1)$$

serves as Heaviside unit step function, i.e.

$$\begin{aligned} \theta(x) &= 1 & x > 0 \\ &= 0 & x < 0. \end{aligned}$$

(Hint: Evaluate (1) by use of the residue theorem, where you should clearly specify the necessary contours.)

B. From the integral representation for $\theta(x)$ derive the oft-quoted relation:

$$\frac{d\theta}{dx} = \delta(x)$$

where $\delta(x)$ is the Dirac delta function.

Work five (5) of the six problems given below. Be sure to work each problem on a separate page since different persons will be grading the examination. The time to work each problem is given in parenthesis.

1. (a) (15 min) An artificial satellite is launched vertically with initial speed \underline{v} . When the vertical speed becomes zero, the satellite is given a transverse speed \underline{u} . Find the resulting orbit in terms of \underline{u} and \underline{v} .
- (b) (15 min) A comet of negligible mass is traveling in a parabolic orbit around the Sun. The orbit is disturbed by a close approach to Jupiter. Assume that during the approach, Jupiter is moving in a straight line and that the comet is subject only to Jupiter's gravitational attraction; also assume that the approach to Jupiter is either of the directions tangent to Jupiter's path. Investigate the conditions under which the velocity of the comet, with respect to the Sun, will be increased or decreased by the encounter.
2. (20 min) A simple plane pendulum is made up of a mass m attached to a string of length ℓ . After the pendulum is set into motion, its length is changed at a constant rate, $d\ell(t)/dt = -\alpha = \text{constant}$. The suspension point remains fixed.
- (a) Calculate the Lagrangian and the Hamiltonian of the system. Is the Hamiltonian equal to the total energy? Are the Hamiltonian or the total energy conserved? Explain.
- (b) Obtain an approximate solution of the angular motion $\theta(t)$ if initially (at $t=0$) $\ell = L$, $\theta = \theta_0 \ll 1$ and $\dot{\theta}(0) = 0$. (Assume α is small).
3. (25 min) A particle of mass m in a symmetric potential $V(x) = A|x|^n$ has a total energy $E = \frac{1}{2} m \left(\frac{dx}{dt} \right)^2 + V(x)$.
- (a) Give a plot of $V(x)$ versus x showing the cases $n = 1/2, 1, 2$, and 3 .
- (b) Solving for dx/dt and integrating, show that the period of the motion is given by

$$T = \frac{2}{n} \left(\frac{2\pi m}{E} \right)^{1/2} \frac{\Gamma(1/n)}{\Gamma\left(\frac{1}{n} + \frac{1}{2}\right)}$$

(properties of beta and gamma function given below)

- (c) Determine T in the limit $n \rightarrow \infty$. How does it compare with the result obtained by directly considering the integrand $(E - A|x|^n)^{-1/2}$ in the same limit? Investigate the behavior of the physical system (potential well) as $n \rightarrow \infty$. You should get a hint from the work of part a.

Hint: the following facts about the beta function $B(p,q)$ and gamma function Γ should be useful:

$$B(p,q) = \int_0^1 dt t^{p-1} (1-t)^{q-1} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad \Gamma(1/2) = \sqrt{\pi}$$

$\Gamma(z)$ has a simple pole at $z = 0$ with the residue one.

4. (a) (10 min) Two pendulums of mass m_1 and m_2 , each of length L , are arranged initially at rest with m_2 hanging vertically and m_1 displaced 25° with the vertical. When released, m_1 collides with m_2 in a completely inelastic collision. Find the maximum angle to which the pendulums move following the collision.
- (b) (15 min) Two particles of mass m and M are initially at rest an infinite distance apart. Show that at any instant their relative velocity of approach due to gravitational attraction is $[2G(M+m)/d]^{\frac{1}{2}}$, where d is the separation at that instant.
5. (15 min) a mass m is connected by a spring to a wall which instantaneously attains a velocity V_0 in the positive x direction. Assume the mass is at rest at $t = 0$, that the spring is not initially stretched, and that the mass slides on a frictionless surface.
- (a) Show that the equation of motion for m is $m\ddot{x}_1 + kx_1 = kV_0 t$, where k is the spring constant and x_1 is measured from the initial equilibrium position of the mass.
- (b) Using the appropriate boundary conditions, solve this equation to find $x_1(t)$.

6. (30 min) The Poisson bracket (P.B.) of two dynamical variables $A(p,q)$ and $B(p,q)$ is defined by

$$\{A,B\} \equiv \sum_k \left\{ \frac{\partial A}{\partial q_k} \frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k} \frac{\partial B}{\partial q_k} \right\}. \quad (1)$$

- (a) Prove the following relation $\{A, H\} = dA/dt$, (2)
where $H(p,q)$ is the Hamiltonian of an N -particle system with $p = (p_1, p_2, \dots, p_{3N})$,
and $q = (q_1, q_2, \dots, q_{3N})$.
- (b) Suppose that G is any function of the coordinates $\{q_i\}$ and momenta $\{p_i\}$ of a dynamical system and ϵ is a small parameter, then the transformation

$$\begin{aligned} p_i' &= p_i + \epsilon \{p_i, G\} \\ q_i' &= q_i + \epsilon \{q_i, G\} \end{aligned} \quad (3)$$

is an infinitesimal canonical transformation generated by G . Suppose that G in (3) is $L_z = x p_y - y p_x$, the z -component of the angular momentum. Write out Eq. (3) explicitly and show that the result is what one expects from an orthogonal transformation.

- (c) It is known that if $F(p_i, q_i)$ is any function of the coordinates and momenta, then the same function of the transformed coordinates is

$$F(p_i', q_i') = F(p_i, q_i) + \{F, G\}. \quad (4)$$

Suppose that the Hamiltonian is invariant under rotation, viz.,

$$H(\vec{r}', \vec{p}') = H(\vec{r}, \vec{p}).$$

By appealing to Eqs. (2) and (4), show that L_z is a constant of the motion.

Work five (5) of the six problems given below. Be sure to work each problem on a separate page since different persons will be grading the examination. The time to work each problem is given in parenthesis.

1. (15 min) Suppose a harmonic oscillator at $t = 0$ is in a state

$$|\psi\rangle = \frac{1}{2} |n=0\rangle + C |n=1\rangle$$

where $E_n = \hbar\omega (n + \frac{1}{2})$ are the energies.

- (a) Given $\langle\psi|\psi\rangle = 1$, find $|C|$.
 (b) Find $|\langle\psi|\psi, t\rangle|^2$. What is the physical significance of this quantity?
 (c) Find $\langle H \rangle_\psi$.
 (d) Do you think the various solutions for C in (a) correspond to different states or are equivalent characterizations of the same state?

2. (25 min) It is well known that the energy eigenvectors of a one-dimensional oscillator with the Hamiltonian $H_0 = p_x^2/2m + \frac{1}{2}m\omega_0^2 x^2$ give rise to the non-vanishing matrix elements

$$\begin{aligned} \langle n | x | k \rangle &= \sqrt{\frac{\hbar}{2m\omega_0}} (k)^{\frac{1}{2}} \text{ for } n = k - 1 \\ &= \sqrt{\frac{\hbar}{2m\omega_0}} (k+1)^{\frac{1}{2}} \text{ for } n = k + 1 \end{aligned} \quad (1)$$

- (a) By use of the closure properties of the energy eigenvectors evaluation

$$\langle n | x^2 | m \rangle. \quad (2)$$

- (b) Suppose that an electron in the harmonic oscillator potential is now perturbed by a constant, uniform electric field \mathcal{E} which points in the x -direction:

$$H' = e\mathcal{E}x \quad (3)$$

Find the energy eigenvalues of the total Hamiltonian $H = H_0 + H'$:

- (i) to first order in H'
 (ii) through the second order in H' .
 (c) Show that the energy eigenvalues of H in part (b) can be obtained exactly. Compare this exact value with the result of part (bii). Discuss.

(turn to the next page)

3. (30 min) The method of partial waves (Faxen-Holtmark) is used to determine the elastic scattering cross section $d\sigma(\theta)/d\Omega$ in quantum mechanics as follows:

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{1}{k^2} \left| \sum_{\ell=0}^{\infty} (2\ell+1) e^{i\delta_{\ell}} \sin\delta_{\ell} P_{\ell}(\cos\theta) \right|^2 \\ &= \frac{1}{k^2} \sum_{\ell=0}^{\infty} \sum_{\ell'=0}^{\infty} (2\ell+1)(2\ell'+1) e^{i(\delta_{\ell}-\delta_{\ell'})} \sin\delta_{\ell} \sin\delta_{\ell'} P_{\ell}(\cos\theta) P_{\ell'}(\cos\theta). \end{aligned}$$

Determine the first two coefficients of the expansion in Legendre polynomials of the elastic scattering cross section $d\sigma/d\Omega$ in terms of the phase shifts.

$$\text{ans: } \frac{d\sigma}{d\Omega} = P_0(\cos\theta) \left[\frac{1}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \sin^2\delta_{\ell} \right] + P_1(\cos\theta) \left[\frac{6}{k^2} \sum_{\ell=0}^{\infty} (\ell+1) \sin\delta_{\ell} \sin\delta_{\ell+1} \cos(\delta_{\ell+1} - \delta_{\ell}) \right]$$

$$\text{Hint: } \int_{-1}^{+1} P_{\lambda}(\mu) P_{\lambda'}(\mu) d\mu = \frac{2}{2\lambda+1} \delta_{\lambda\lambda'} \quad ; \quad \cos\theta P_{\ell} = \frac{(\ell+1)P_{\ell+1} + \ell P_{\ell-1}}{(2\ell+1)}$$

Expand $d\sigma/d\Omega = \sum_{\lambda} a_{\lambda} P_{\lambda}(\cos\theta)$ and use orthogonality of P_{λ} 's.

4. (15 min) Consider a particle subject to a one-dimensional delta-function potential:

$$V(x) = -V_0 \delta(x),$$

where V_0 is a positive constant. ~~Units~~ ^{Units} are chosen such that $\hbar = 1$ and the particle mass is $m = 1$. For negative energies, define

$$E = -\frac{1}{2} K^2 < 0.$$

- (a) Write down the general form of the particle's wavefunction, ψ_K , appropriate to $E < 0$.
- (b) Use the matching conditions at $x = \pm a$ to show that there exists a single bound state. What is its energy?

(turn to the next page)

5. (35 min) Consider the three Pauli spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(a) Show that in the basis in which σ_z is diagonal the functions

$$|v_+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad |v_-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

are eigenfunctions of the matrix σ_x .

(b) Find the unitary matrix U which transforms these functions to

$$|v_+' \rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |v_-' \rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and obtain the Pauli spin matrices in this new basis.

6. (35 min) An electron moves in the presence of a uniform magnetic field in the z-direction ($\vec{B} = B \hat{z}$)

(a) Evaluate the commutator $[\pi_x, \pi_y]_-$, (1)

$$\text{where } \pi_x = p_x - eA_x/c, \quad \pi_y = p_y - eA_y/c \quad (1a)$$

(Choose the gauge $A_x = -Ay$, $A_y = Ax$, $A_z = 0$, and show that these give the correct \vec{B} field.)

(b) Construct the Hamiltonian \mathcal{H} of the system. By comparing \mathcal{H} and commutator (1a) with those of the one-dimensional oscillator problem, show that one can immediately obtain the (Landau) energy levels:

$$E(p_z, n) = \frac{\hbar^2 \beta^2}{2m} + \hbar \omega_0 \left(n + \frac{1}{2} \right) \quad (2)$$

where $\hbar\beta$ are the continuous eigenvalues of the p_z . What would be the value of ω_0 ?

(c) Argue that the wave function corresponding to \mathcal{H} can be put in the form

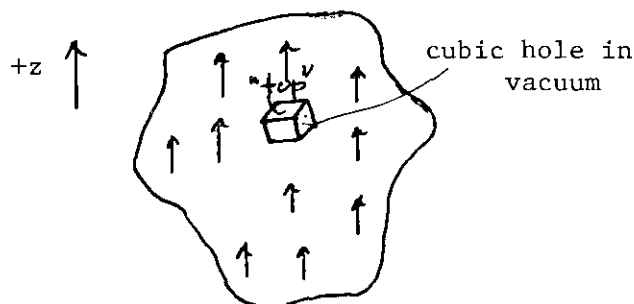
$$\psi(x, y, z) = e^{i(\alpha x + \beta z)} f(y). \quad (3)$$

What would be the nature of $f(y)$? Eq. (3) shows that the motion along the z-direction is unbounded, which is reasonable because it is parallel to the external \vec{B} field. There exists an apparent asymmetry between the motion along the x-direction and that along the y-direction, although both directions are perpendicular to the \vec{B} field. Resolve this difficulty.

Hint: Notice that the energy levels do not depend on $\hbar\alpha$, hinting at an infinite degeneracy.

Work five (5) of the six problems given below. Be sure to work each problem on a separate page since different persons will be grading the examination. The time to work each problem is given in parenthesis.

1. (20 min) A cubic hole is cut in a piece of material which has a uniform electric polarization: $\vec{P} = P_0 \hat{z}$:



- (a) Taking the origin of coordinates at the center of the cubic hole, show that the electric field at the origin can be expressed as

$$\vec{E}(0) = 2 P_0 \int \frac{\vec{x}'}{r'^3} da'$$

where the integration is over the "top" of the cube, as shown above.

- (b) Using symmetry and the concept of solid angle, argue now that this expression reduces to

$$\vec{E}(0) = \frac{4\pi}{3} P_0 \hat{z} .$$

2. (30 min) A large parallel-plate capacitor is made up of two plane conducting sheets, one of which has a small hemispherical boss of radius a on its inner surface. The conductor with the boss is kept at zero potential, and the other conductor is at a potential such that far from the boss the electric field between the plates is E_0 .

- (a) Calculate the surface-charge densities at an arbitrary point on the plane and on the boss, and sketch their behavior as a function of the distance (or angle).
- (b) Show that the total charge on the boss has the magnitude $3E_0 a^2/4$. How much is this larger than the surface charge on the diametric plate, i.e., that part without the boss?



3. (15 minutes) Consider a dipole of moment \mathbf{p} located at the center of a dielectric sphere of radius R and permittivity ϵ_1 . The permittivity of the material external to the sphere is ϵ_2 .

(i) Show that the potential inside the dielectric sphere can be expressed in the form

$$\Phi_1(r, \theta) = \frac{p}{4\pi\epsilon_1} \left[\frac{\cos \theta}{r^2} + \frac{2(\epsilon_1 - \epsilon_2)}{R^3(\epsilon_1 + 2\epsilon_2)} r \cos \theta \right] \quad r \leq R$$

(ii) Show that the potential outside the dielectric sphere can be expressed in the form

$$\Phi_2(r, \theta) = \frac{3p}{4\pi(\epsilon_1 + 2\epsilon_2)} \left(\frac{\cos \theta}{r^2} \right) \quad r \geq R$$

(iii) What does the above reduce to in the limit as ϵ_1 approaches ϵ_2 ?

4. (40 min) The equation of the surface of a conductor is $r = a(1 + \epsilon P_n(\cos \theta))$, where $\epsilon \ll 1$, and $P_n(\mu)$ is the Legendre polynomial of degree n . Show that if the conductor is placed in a uniform E field parallel to the polar axis the surface charge density is

$$\sigma = \sigma_0 + \frac{\epsilon E}{4\pi} \frac{3n}{2n+1} [(n+1) P_{n+1}(\cos \theta) + (n-2) P_{n-1}(\cos \theta)] ,$$

where σ_0 is the induced charge density for $\epsilon = 0$.

Hint: Recall that when $\epsilon = 0$, the potential is given by

$$\Phi_0 = E r \cos \theta - \frac{E a^3}{r^2} \cos \theta = E r \left(1 - \frac{a^3}{r^3} \right) \mu , \quad \text{with } \mu = \cos \theta = P_1(\mu).$$

Write the distorted potential in the form

$$\Phi(r, \theta) = E r \mu + \frac{A}{r^2} \mu + \frac{B(\epsilon)}{r^n} P_{n-1}(\mu) + \frac{C(\epsilon)}{r^{n+2}} P_{n+1}(\mu),$$

where the constants B and C are of the order of ϵ , and can be determined in part from the boundary condition. The following recursion relation

is used more than once:

$$\mu P_n(\mu) = \frac{n}{2n+1} P_{n-1}(\mu) + \frac{n+1}{2n+1} P_{n+1}(\mu) .$$

5. (20 min) A cubical volume with sides L is centered at the coordinate origin and is aligned with the x, y, z axes. Within the volume is a charge density $\rho = K x$.
- (a) Calculate the monopole, dipole and all the quadrupole moments of this charge distribution.
- (b) What is the form of the electric field far away from the cube?
6. (20 min) Using the electromagnetic field tensor $F_{\mu\nu}$ and the transformation matrix Λ , explicitly derive the field expressions for a moving frame in terms of the fields in a stationary frame.

$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & -i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \quad F_{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1/c \\ -B_3 & 0 & B_1 & -iE_2/c \\ B_2 & -B_1 & 0 & -iE_3/c \\ iE_1/c & iE_2/c & iE_3/c & 0 \end{pmatrix}$$

Work five (5) of the six problems given below. Be sure to work each problem on a separate page since different persons will be grading the examination. The time to work each problem is given in parenthesis.

1. (25 min) A classical one-dimensional harmonic oscillator described by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 \quad (1)$$

is in thermal contact with a heat bath at temperature T .

- (a) Calculate the partition function Z_c in the canonical ensembles. Find the explicit values of $\langle E \rangle$ and $\langle (E - \langle E \rangle)^2 \rangle$.
- (b) Calculate the quantal partition function for the one-dimension oscillator of Eq. (1). Find $\langle E \rangle$, and compare with the corresponding result obtained in part (a). Discuss.

2. (25 min) (a) The frequency of an electromagnetic mode of wave vector \vec{k} is $\omega_{\vec{k}} = c k$. Show that in a box of volume V there will be

$$2 \frac{V d^3k}{(2\pi)^3} \quad (1)$$

such modes in a region d^3k surrounding a given wave vector. The Hamiltonian of the system is $\sum_{\vec{k}} \hbar \omega_{\vec{k}} n_{\vec{k}}$, where $n_{\vec{k}}$ is the number of excitations in the mode with wave vector \vec{k} . Show that the Helmholtz free energy of the electromagnetic radiation in the cavity is

$$A = \frac{V k_B T}{c^3} \int_0^{\infty} d\omega \omega^2 \ln [1 - e^{-\beta \hbar \omega}] , \quad \beta = (k T)^{-1}.$$

- (b) By comparing the expressions $P = -\partial A / \partial V$ and $E = \partial(\beta A) / \partial \beta$ for the pressure and internal energy, respectively, show that

$$PV = E/3.$$

3. (15 min) Determine the threshold energy of producing antiprotons by collisions of a beam of protons incident on protons in a stationary target. Take the rest mass energy of a proton to be 938 MeV.

(Please turn to the next page)

4. (35 min) The intermolecular potential energy $U(r_1, r_2, \dots, r_N)$ of a real gas of N particles is a homogeneous function of degree m in the position coordinates of the particles:

$$U(\lambda \vec{r}_1, \lambda \vec{r}_2, \dots, \lambda \vec{r}_N) = \lambda^m U(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) . \quad (1)$$

Show that the equation of state is of the form

$$P T^{-1+3/m} = f(V T^{-3/m}) \quad (2)$$

where f is an undetermined function of one variable.

Hint: Argue that in the partition function only the configurational integral

$$Q_N = \int \dots \int d\vec{r}_1 d\vec{r}_2 \dots d\vec{r}_N \exp (-U(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) / k_B T)$$

need to be considered to calculate the pressure. Work with $T^{-3N/m} Q_N(V, T)$

and exploit the fact that the function is invariant under the substitutions:

$$T \rightarrow T \lambda^m , \quad V \rightarrow V \lambda^3 .$$

5. (25 min) Find an expression for the differential scattering cross section and an expression for the total cross section of elastic scattering by a ball of radius R .

6. (30 min) (a) Find the limit of

$$\lim_{\alpha \rightarrow 0} \frac{1}{\alpha} \left[\frac{1}{|x|^\alpha} - 1 \right] .$$

- (b) The Gamma function is defined by

$$\Gamma(z) = \int_0^\infty dt e^{-t} t^{z-1} = U(x,y) + i V(x,y) \quad (1)$$

where $z = x + i y$.

(i) Find the real and imaginary parts of $\Gamma(z)$.

(ii) On the basis of the above result show that $\Gamma(z^*) = \Gamma^*(z)$.

- (c) Evaluate

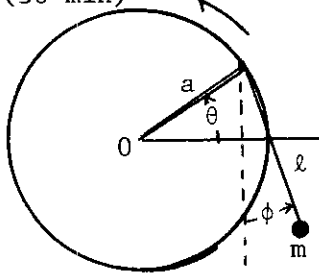
$$W = i^i .$$

Note that W is multivalued because $\ln i$ is so.

Work five (5) of the six problems given below. Be sure to work each problem on a separate page since different persons will be grading the examination. The time to work each problem is given in parenthesis.

(30 min)

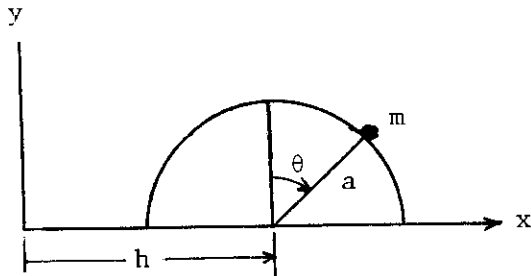
1.



A simple pendulum of length l (mass of rod = 0), with mass point m has a radius of suspension that rotates with angular velocity $\dot{\theta}$ on the vertical circle of radius a as shown in the figure to the left. All the motion is confined to a single, fixed vertical plane.

- (a) Set up the Lagrangian in terms of $\theta, \phi, \dot{\theta}, \dot{\phi}$. (Use 0 as the reference point for the potential energy).
- (b) Use the matrix method to find the Hamiltonian in terms of $p_\theta, p_\phi, \theta, \phi$.
- (c) Write down Hamilton's four equations using the Hamiltonian from part (b).
- (d) If $\dot{\theta} = \omega = \text{constant}$, find the new Hamiltonian in terms of p_ϕ, ϕ, θ , and ω and find the two Hamilton equations involving ϕ and p_ϕ .

2. (30 min)

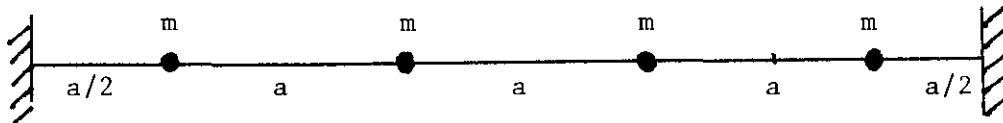


A particle of mass m starts from rest on top of a smooth hemisphere of radius "a." The hemisphere, of mass M , moves without friction along a horizontal surface. By using conservation of energy and momentum and by applying the approximation $M \gg m$ consistently, show that the angle at which the particle loses contact with the hemisphere is given by

$$\cos \theta \cong \frac{2}{3} + C \frac{m}{M}.$$

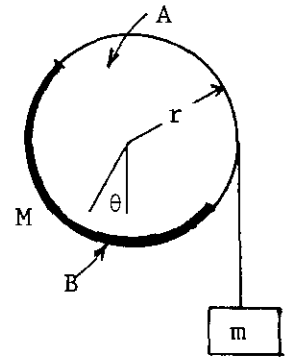
Find the value of the constant "C."

- 3. (40 min) When placed under tension T , a rubber band with four equal masses m placed as shown has a length $4a$. Assume the length of the rubber band is zero under vanishing tension. Assuming only longitudinal displacements of the masses, calculate the resonant frequencies and normal modes of the system. Sketch each normal mode.



4. (20 min) The pulley A turning smoothly on a horizontal axle is light but has a heavy semicircular rim B of mass M. A string passing around the pulley carries a mass m. Draw a graph showing how the frequency of small oscillations of m depends on the value of m. Make sure you understand why there is a critical mass m_c at which the behavior changes. What happens when $m > m_c$? Show that near the critical point,

$$\omega_0 \propto (m_c - m)^{\frac{1}{2}}. \quad (1)$$



5. (30 min) Given a central force $\frac{d\vec{p}}{dt} = f(r) \frac{\vec{r}}{r}$, where \vec{p} is the linear momentum of a particle of (reduced) mass m.

(a) Establish the relation $\frac{d}{dt} (\vec{p} \times \vec{L}) = -m f(r) r^2 \frac{d}{dt} \left(\frac{\vec{r}}{r} \right)$, (2)
where $\vec{L} = \vec{r} \times \vec{p} = m \vec{r} \times \dot{\vec{r}}$.

- (b) Suppose that the force is an inverse-square law: $f(r) = -k/r^2$. Eq. (2) then shows that the following Runge-Lenz vector is constant in time:

$$\vec{R} = \vec{p} \times \vec{L} - m k \frac{\vec{r}}{r}. \quad (3)$$

Show that $\vec{R} \cdot \vec{L} = 0$ (4)

Furnish the geometrical and physical interpretation of the above result. If θ is the angle between the fixed constant vector \vec{R} and \vec{r} , establish the following equation of a conic section

$$\frac{1}{r} = \frac{mk}{\ell^2} \left(1 + \frac{R}{mk} \cos\theta \right), \quad (5)$$

where $\ell^2 = \vec{L} \cdot \vec{L}$. What is the geometric meaning of R/mk ? Give the physical meaning of \vec{R} in the Kepler problem, i.e., $k = GM$, where G is the universal gravitational constant, and M is the mass of the heavier body.

- (c) It is generally the case that the existence of a constant of motion implies that of the corresponding symmetry group of the Hamiltonian, and often the degeneracy of the energy spectrum. For the bound Kepler problem the group is identified with $SO(4)$ or $R(4)$, the Lie group of four-dimensional real rotation. Discuss the pertinent points for the hydrogen Kepler problem, i.e., $k = e^2$, where e is the proton charge.

(examination continued next page, please)

6. (30 min) (a) Find the Lagrangian for a simple pendulum of mass m placed in a uniform gravitational field whose point of support moves uniformly on a vertical circle with constant frequency γ .
- (b) Find the Lagrangian for a simple pendulum of mass m placed in a uniform gravitational field whose point of support oscillates horizontally in the plane of motion of the pendulum according to the law $x = a \cos \gamma t$.
- (c) Find the Lagrangian for a simple pendulum of mass m placed in a uniform gravitational field whose point of support oscillates vertically according to the law $y = a \cos \gamma t$.
- (d) Use the results which you obtained above to determine the positions of stable equilibrium of a pendulum whose point of support oscillates vertically with a high frequency γ ($\gg \sqrt{g/\ell}$).
- (e) Again use the results which you obtained above to determine the positions of stable equilibrium of a pendulum whose point of support oscillates horizontally with a high frequency γ ($\gg \sqrt{g/\ell}$).

Work five (5) of the six problems given below. Be sure to work each problem on a separate page since different persons will be grading the examination. The time to work each problem is given in parenthesis.

1. (30 min) Define the linear momentum operator $p_r = \frac{\hbar}{i} \left(\frac{\partial}{\partial r} + \frac{1}{r} \right)$ in spherical coordinates.

(a) Use $\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$, $p_x = \frac{\hbar}{i} \frac{\partial}{\partial x}$, $p_y = \frac{\hbar}{i} \frac{\partial}{\partial y}$, $p_z = \frac{\hbar}{i} \frac{\partial}{\partial z}$, and the chain rule of differentiation to show that

$$\frac{\vec{r} \cdot \vec{p}}{r} = \frac{\hbar}{i} \frac{\partial}{\partial r}.$$

(b) Write the following term in rectangular components, perform the necessary differentiation, and then transfer back to spherical coordinates to show that

$$\vec{p} \cdot \frac{\vec{r}}{r} = \frac{\hbar}{i} \left(\frac{2}{r} + \frac{\partial}{\partial r} \right).$$

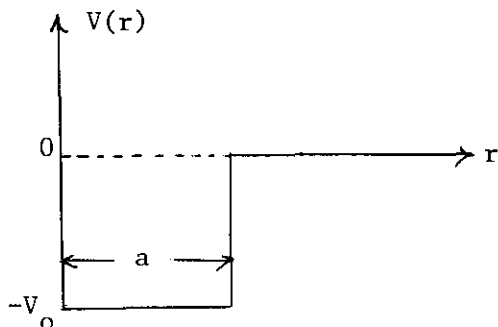
Add parts (a) and (b) together to write the alternate form $p_r = \frac{1}{2} \left(\vec{p} \cdot \frac{\vec{r}}{r} + \frac{\vec{r} \cdot \vec{p}}{r} \right)$.

(c) Find the expectation value $\langle p_r \rangle$ for the hydrogen-atom wave function ($a_B = \text{Bohr radius}$)

$$\psi_{210}(r, \theta, \phi) = \frac{1}{2\sqrt{2\pi}} \left(\frac{1}{a_B} \right)^{3/2} \frac{1}{2a_B} \left(r e^{-r/(2a_B)} \right) \cos\theta.$$

Hint: Use $p_r = \frac{\hbar}{i} \left(\frac{\partial}{\partial r} + \frac{1}{r} \right)$.

2. (30 min) A particle of mass m moves in a potential $V(r) = -V_0$ when $r < a$, and $V(r) = 0$ when $r > a$. Find the least value of V_0 such that there is a bound state of zero energy and zero angular momentum.



Explain physically why the lowest bound state has zero angular momentum.

3. (20 min) A particle of mass m moves in a potential well of the form $V(x) = \lambda x^4$. Use the WKB approximation to determine the energy eigenstates E_n . Express your answer in terms of the Beta function:

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}, \quad [m,n > 0].$$

4. (30 min) The Hamiltonian of an atom in a uniform magnetic field may be partially described by

$$H = \alpha \vec{I} \cdot \vec{J} + (g_I \beta_I \vec{I} + g_J \beta_J \vec{J}) \cdot \vec{B},$$

where \vec{B} is the magnetic field, \vec{I} and \vec{J} are the nuclear and electronic angular momentum operators, and β_I and β_J are nuclear and Bohr magnetons (constants).

Choose appropriate sets of good quantum numbers and evaluate the energy levels to first order in the small quantities when

- (a) $|\beta_I \vec{B}|$ and $|\beta_J \vec{B}| \gg \alpha$, and
 (b) $|\beta_I \vec{B}|$ and $|\beta_J \vec{B}| \ll \alpha$.

5. (30 min)--(15 min each part):

A. A particle is confined to a one-dimensional box with infinite walls at $(0,L)$. If initially the particle is in an eigenstate of the Hamiltonian H of the system its state vector at $t = 0$ is given by

$$\psi_n(x,0) = \phi_n(x) = (2/L)^{1/2} \sin(n\pi x/L). \quad (1)$$

- (a) Suppose, on the other hand, the initial state is not an eigenstate of H , then it is still expandable in terms of $\{\phi_n\}$:

$$\psi(x,0) = \sum_n b_n \phi_n(x), \quad \text{with } b_n = \langle \phi_n | \psi(x,0) \rangle. \quad (2)$$

Show that the state vector for $t > 0$ is then given by

$$\psi(x,t) \equiv e^{-iHt/\hbar} \psi(x,0) = \sum_n b_n e^{-i\omega_n t} \phi_n(x), \quad (3)$$

$$\text{where } \hbar \omega_n = E_n = n^2 E_1.$$

- (b) Suppose that the initial state is given by $\psi(x,0) = \sqrt{\frac{2}{L\sqrt{5}}} \left\{ \sin \frac{2\pi x}{L} + 2 \sin \frac{\pi x}{L} \right\}$. (4)

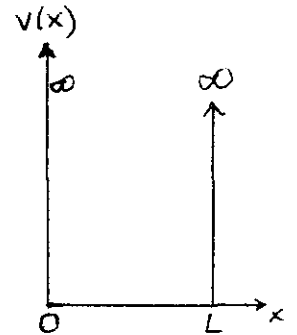
The energy of the system is measured at $t > 0$. Find the expectation value

$$\langle E \rangle_{t>0} \text{ of the energy at } t > 0 \text{ for the initial state (4).}$$

B. Consider a particle in a one-dimensional system whose Hamiltonian is given by

$$H = \frac{p^2}{2m} + V(x).$$

- (a) Evaluate the commutator $[H,x]$ and the double commutator $[[H,x],x]$.
 (b) Availing yourself of the above results, prove $\sum_\beta \frac{2m}{\hbar} |\langle \alpha | x | \beta \rangle|^2 \omega_{\alpha\beta} = 1$, (1)
 where $\hbar \omega_{\alpha\beta} = E_\alpha - E_\beta$ and $|\alpha\rangle$ is an energy eigenket with eigenvalue E_α .
 What is $(2m/\hbar) \sum_\beta |\langle \alpha | x | \beta \rangle|^2 \omega_{\alpha\beta}$ called? What is relation (1) called?



6. (30 min) The following problem deals with the free-particle propagator.

(i) Starting with the Gaussian wave packet whose initial state is given by

$$\Psi(x,0) = \frac{1}{\sqrt{a}(2\pi)^{\frac{1}{4}}} \exp[ik_0x] \exp\left[-\frac{x^2}{4a^2}\right]$$

show that the state of the particle at any time $t > 0$ is given by

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} b(k) \exp[i(kx - \omega t)] dk$$

(ii) Using the result obtained in (i), derive the explicit form for the free-particle propagator, $K(x', x; t)$. (Hint: To obtain a convergent integral, first replace i by $\alpha = i + \epsilon$, where ϵ is a small positive real number. After integrating, let ϵ go to zero.) Besides the free-particle propagator, what is $K(x', x; t)$ also often called?

(iii) Give the physical interpretation for the free-particle propagator. What is the free-particle propagator used for? Why is it so important?

(iv) Using the result you obtained in (ii) and the formal prescription given below

$$\Psi(x, t) = \int_{-\infty}^{+\infty} \Psi(x', 0) K(x', x; t) dx'$$

complete the integration and find the explicit $\Psi(x,t)$ for the given problem. What is the corresponding probability density?

(v) Where does the equation given in part (iv) come from and why are we using it in this situation?

(vi) Compare your result from (iv) with the initial probability density and state the modifications (if any) to the generic shape of $P(x, 0)$ as time progresses.

(vii) If you represent a piece of chalk by a wave packet with $a \simeq 1$ cm and $m \simeq 1$ gm calculate the time interval t after which the packet will begin to distort significantly. Use your result to explain the fact that classical objects are never observed to suffer quantum mechanical spreading.

Work five (5) of the six problems given below. Be sure to work each problem on a separate page since different persons will be grading the examination. The time to work each problem is given in parenthesis.

1. (20 min) A thin copper ring rotates freely about a diameter, which is perpendicular to a uniform magnetic field B . Its initial frequency of rotation is ω . Calculate the time taken for the frequency to decrease to $1/e$ of its original value, under the assumption that the energy dissipates by Joule heating. (The resistivity of copper is $1.8 \times 10^{-8} \Omega \text{ m}$; the density of copper is $8.9 \times 10^3 \text{ kg/m}^3$. Take $B = 2 \times 10^{-2} \text{ T}$).

2. (15 min) The time-average potential of a neutral hydrogen atom is given by

$$\phi(r) = q \frac{e^{-\alpha r}}{r} \left(1 + \frac{\alpha r}{2} \right) \quad (\text{A})$$

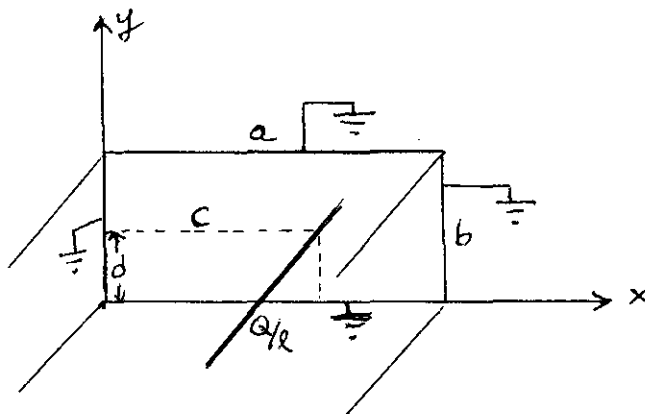
$$= \frac{q}{r} + q \left(\frac{e^{-\alpha r} - 1}{r} + \frac{\alpha}{2} e^{-\alpha r} \right) \quad (\text{B})$$

where q is the magnitude of the electronic charge and $1/\alpha = a_0/2$, a_0 being the Bohr radius. Find the distribution of charge (both discrete and continuous) which will give this potential, and give the physical interpretation of your result.

Caution: If you deal with the potential ϕ given by (A) you will meet mathematical and physical ambiguities. Discuss why the expression (B) is free of such ambiguities. It may be of interest to you that the wave function of the $1s$ state of a hydrogen atom is given by

$$\psi_{1s}(r) = \frac{1}{\sqrt{\pi a_0^3}} e^{-\alpha r/2} .$$

3. (35 min) Find the electrostatic potential $V(x,y)$ inside a prism whose walls are given by $x = 0$, $x = a$ and $y = 0$, $y = b$. A line charge of strength Q/ℓ lies at $x = c$, $y = d$, where $0 < c < a$ and $0 < d < b$. The line charge is parallel to the walls at $x = 0$, $x = a$, and the walls at $y = 0$, $y = b$. The four walls are all grounded. Please use SI (mks) units, not cgs units!



4. (40 min) The general solution for the two-dimensional potential problem with the potential specified on the surface of a cylinder of radius b is given by a series form:

$$\Phi(\rho, \phi) = a_0 + b_0 \ln \rho + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi + \alpha_n) + \sum_{n=1}^{\infty} \frac{b_n}{\rho^n} \sin(n\phi + \beta_n) \quad (1)$$

- (a) Restrict yourself to an interior Dirichlet problem, where no charge resides inside the cylinder. Determine the acceptable series solution by discussing the fate of the constants a_n , b_n , α_n , and β_n for $n \geq 0$ to satisfy the boundary conditions imposed on the system.
- (b) The series solution given in part (a) can be summed in a closed form, as Poisson did, by making use of the relation

$$\frac{1}{2} + \sum_{n=1}^{\infty} y^n \cos nx = \frac{1}{2} \frac{1 - y^2}{1 - 2y \cos x + y^2} \quad (2)$$

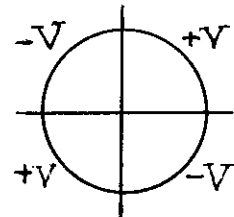
Establish Eq. (2) by expressing $\cos nx$ in exponential forms by use of Euler's relation, and summing the infinite geometric series.

- (c) Avail yourself of Eq. (2) to obtain the potential inside the cylinder in the form of the Poisson integral

$$\Phi(\rho, \phi) = \frac{1}{2\pi} \int_0^{2\pi} \Phi(b, \phi') \frac{b^2 - \rho^2}{b^2 + \rho^2 - 2b\rho \cos(\phi' - \phi)} d\phi' \quad (3)$$

Suppose that a long hollow conducting cylinder of radius b is divided into equal quarters, alternate segments being held at potential $+V$ and $-V$. By making use of the series solution obtained in part (a), show that the potential inside the cylinder is

$$\Phi(\rho, \phi) = \frac{4V}{\pi} \sum_{n=0}^{\infty} \left(\frac{\rho}{b}\right)^{2(2n+1)} \frac{\sin 2(2n+1)\phi}{2n+1} \quad (4)$$



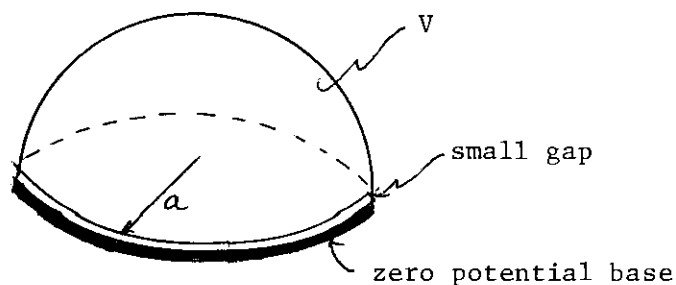
Hint: Exploit the symmetries of $\Phi(b, \phi)$ under $\phi \rightarrow \phi + \pi$ and $\phi \rightarrow \phi + \pi/2$.

- (d) Sum the series in (4) and show that $\Phi(\rho, \phi) = \frac{2V}{\pi} \tan^{-1} \left(\frac{2b^2 \rho^2 \sin 2\phi}{b^4 - \rho^4} \right)$. (5)

$$\text{Recall that } \text{Im} \ln \frac{1+z}{1-z} = \tan^{-1} \left(\frac{2 \text{Im } z}{1 - |z|^2} \right) \quad (6)$$

5. (20 min) A hemispherical metal cap of radius " a " is maintained at potential V , while its base (a plane surface) is held at zero potential.

- (a) Find the potential, $\Phi(\vec{x})$, inside the hemisphere. (You may leave some results in integral form, if necessary).
- (b) Evaluate the electric field, \vec{E} , just above the center of the base of the hemisphere.



6. (30 min) (a) Derive Gauss's law for time-independent fields.
- (b) State Maxwell's equations for static conditions (time-independent fields).
- (c) Show explicitly that Ampere's law as given in part (b) is valid only for steady-state conditions and is insufficient for the case of time-dependent fields. (Hint: Prove that it is incompatible with the assumption of conservation of charge).
- (d) Make Maxwell's famous substitution and obtain the modified form of Ampere's law. Show explicitly where Maxwell came up with this substitution in the first place. (Hint: Use the equation of continuity and Gauss's law).
- (e) Show that the equation you obtained in part (d) is compatible with conservation of charge.
- (f) What is the physical significance of the displacement current? Why is it central to the theory of electromagnetic wave radiation? What does it (along with Faraday's law) say about \vec{E} and \vec{B} fields?
- (g) Working backwards from the Maxwell-Ampere law (which you may assume to have been experimentally determined) show that Gauss's law holds for time-dependent fields. Be sure to explain the conditions which must be met by the field at sometime in the past for the above to hold true.
- (h) State Maxwell's equations for time-dependent electromagnetic fields.

Part IV

Statistical Mechanics/General

Work five (5) of the six problems given below. Be sure to work each problem on a separate page since different persons will be grading the examination. The time to work each problem is given in parenthesis.

1. (25 min) The potential $V(r)$ between the two atoms ($m_H = 1.672 \times 10^{-24}$ gm) in a hydrogen molecule is given by the empirical expression

$$V(r) = D \left(e^{-2a(r-r_0)} - e^{-a(r-r_0)} \right), \quad (1)$$

where r is the distance between the atoms and $D = 7 \times 10^{-12}$ erg, $a = 2 \times 10^8$ cm $^{-1}$, and $r_0 = 8 \times 10^{-9}$ cm.

- (a) Estimate the temperature at which rotation (T_R) and vibration (T_V) begin to contribute to the specific heat of hydrogen gas.

Hint: Expand $V(r)$ in a power series of $(r - d)$, where $r = d$ is the equilibrium distance. This will supply you Hooke's constant k and hence, ω , the frequency of the radial vibration.

- (b) Give the approximate values of C_V and C_p for the following temperatures:

$$T_1 = 25 \text{ K}, \quad T_2 = 250 \text{ K}, \quad T_3 = 2500 \text{ K}, \quad \text{and} \quad T_4 = 10,000 \text{ K}.$$

Neglect ionization and dissociation.

2. (35 min) Consider a system of noninteracting fermions, where the single particle states are labeled by a wave vector \vec{k} and a spin index σ , collectively denoted by i , and given by:

$$\langle n_i \rangle = \frac{1}{e^{\beta(\epsilon_i - \mu)} + 1}, \quad (1)$$

where $\beta = 1/kT$. Similarly, for a system of noninteracting massive bosons, the mean number of particles occupying state $|i\rangle$ is given by

$$\langle n_i \rangle = \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1}. \quad (2)$$

- (a) Show that for bosons the chemical potential can never be positive, i.e., $\mu \leq 0$, always.
 (b) Under what condition will the above quantum distributions approach the classical Maxwellian distribution? Hint: You have to establish an inequality for the fugacity $\exp(\beta\mu)$ by considering Boltzmann's expression for N :

$$N = \frac{V}{h^3} e^{\beta\mu} 4\pi \int_0^\infty dp p^2 e^{-\beta p^2/2m} \quad \text{to arrive at } n\lambda^3 = e^{\beta\mu}, \quad \text{where } n = N/V, \text{ and}$$

λ is the deBroglie thermal wavelength.

Discuss the significance of the inequality in terms of n , m , and T . Recall the value of the integral $\int_{-\infty}^{+\infty} e^{-\alpha p^2} dp = \sqrt{\pi/\alpha}$. If the integrand involves an even power of p , it can be evaluated by a parametric differentiation of the above result.

- (c) The expectation value of the number of bosons is given by summing $\langle n_i \rangle$ for all states $|i\rangle$:

$$N = \sum_i \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1}.$$

When the state index refers to the continuum it is necessary to replace the summation by an integral. Justify the following rule of replacement:

$$\sum_i = g \frac{V}{h^3} \int 4\pi p^2 dp, \quad (3)$$

where g is the weight factor, e.g., polarizations, or if the particle has a spin s , then $g = 2s+1$, and V is the volume of the system.

(problem continued next page, please)

2. (continued)

(d) Now consider a system of noninteracting fermions with single-particle energy $\epsilon = p^2/2m$. Let $D(\epsilon)$ denote the density of single-particle states, defined by

$$D(\epsilon)d\epsilon = g \frac{V}{h^3} 4\pi p^2 dp. \quad (4)$$

Find $D(\epsilon)$, and plot it as a function of ϵ .

(e) Derive a formula for the maximum kinetic energy ϵ_F of an electron in a non-interacting Fermi gas containing N electrons in a volume V at absolute zero temperature and show that

$$\epsilon = \epsilon_{\max} = \frac{h^2}{2m} \left(\frac{3N}{8\pi V} \right)^{2/3}, \quad (5)$$

where ϵ_F is also known as the Fermi energy.

3. (30 min) For a fluid of one species of molecules, Van der Waals derived the equation of state:

$$\left(P + \frac{N^2 a}{V^2} \right) (V - N b) = N k T,$$

where P is the pressure, V is the volume, T the (absolute) temperature, N the number of molecules, and k is Boltzmann's constant. With properly chosen a and b this equation, improved by Maxwell's equal area construction, gives a good fit to the observed values of P , V , and T , and a semiquantitative description of condensation and the critical point.

(a) What are the physical properties of molecules that are approximately taken into account by making a and b different from zero?

(b) Find the critical pressure P_c , critical volume V_c , and critical temperature T_c as a function of N , k , a , and b . Also, determine the value of $P_c V_c / NkT_c$.

4. (30 min) A slightly non-ideal gas has the equation of state (assumed to hold at all temperatures and volumes):

$$P = kT \left(\frac{N}{V} + \left(\frac{N}{V} \right)^2 B(T) \right),$$

where P = pressure, V = volume, and N is the total number of molecules.

(a) Given that $C_V \equiv (dQ/dT)_T$ for an ideal gas is $(3/2) N k$, show for the above equation of state that

$$C_V = N k \left[3/2 - (N/V) \frac{d}{dT} \left(T^2 \frac{dB}{dT} \right) \right].$$

[Hint: Consider $(\partial U/\partial V)_T$, where U is the internal energy].

(b) Determine the entropy, S , of this gas given that $S = (3/2) Nk \ln T + kN \ln V + \text{constant}$ for an ideal gas.

5. (20 min) Evaluate the integral

$$\int_0^{\infty} \frac{\sin^2 x}{x^2} dx.$$

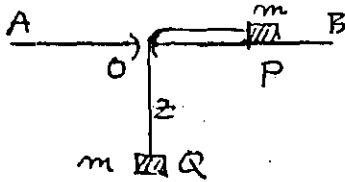
Justify each step.

6. (30 min)

- (a) Two identical ideal gases at the same pressure P and containing the same number of particles N but at different temperatures T_1 and T_2 are in vessels with volumes V_1 and V_2 . The vessels are then connected. Find the change in entropy.
- (b) Find the work done on an ideal gas in an adiabatic compression.
- (c) Find the quantity of heat gained by a gas in an isochoric process, i.e., one which occurs at constant volume.
- (d) Find the work done and quantity of heat gained in an isobaric process, i.e., one which occurs at a constant pressure.
- (e) Find the work done on a gas and the quantity of heat which it gains in compression from volume V_1 to V_2 in accordance with the equation $PV^n = a$ (this is known as a polytropic process).
- (f) Find the work done on an ideal gas and the quantity of heat which it gains in going through a cyclic process (i.e., one in which it returns to its initial state at the end of the process), consisting of two isochoric and two isobaric processes: the gas goes from a state with pressure and volume P_1, V_1 to states $P_1, V_2; P_2, V_2; P_2, V_1; P_1, V_1$ again.

Work five (5) of the six problems given below. Be sure to work each problem on a separate page since different persons will be grading the examination. The time to work each problem is given in parenthesis. Distribute your time wisely. Spend no more than 30 minutes on each problem. If you haven't finished a problem, then go on to the next one.

1. (30 min)



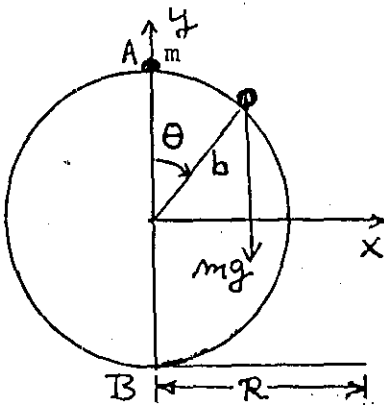
AB represents a frictionless horizontal plane having a small opening at O. A string of length l passes through O and has at its ends a particle P of mass m and a particle Q of equal mass m which hangs freely. The particle P is given an initial velocity of magnitude v_0 at right angles to string OP when the length of string OP is a .

- (a) Set up the Lagrangian of the system using cylindrical coordinates and reduce the Lagrangian to the variables r and θ by use of the constraint.
- (b) Write a differential equation for the motion of P in terms of distance $OP = r$ (Express your answer in terms of r , a , and v_0).
- (c) Show that the value of \ddot{r} is given by

$$\ddot{r} = \left[g \left(\frac{a}{r} - 1 \right) + \frac{v_0^2}{2} \left(1 - \frac{a^2}{r^2} \right) \right]^{1/2}$$

- (d) Prove that in order for the particle P to remain in stable equilibrium in a circle of radius a , then v_0 must be equal to \sqrt{ag} , and that if it is displaced slightly from equilibrium, it will oscillate with simple harmonic motion with period $T = 2\pi\sqrt{(2a)/(3g)}$.

2. (30 min)



A particle of mass m rests at the top of a sphere of radius b as shown in the figure to the left. The particle is displaced slightly to the right so that it slides without friction and without rolling.

- (a) Find the angle θ at which the particle leaves the sphere.
- (b) What is the speed v when it leaves the sphere? (Express your answer in terms of m and g).
- (c) Find the horizontal distance R from the base point B of the sphere at which the mass lands on the plane.

$$\text{Ans. } (5\sqrt{5} + 20/2) \frac{b}{27}$$

3. (30 min) Using Newton's expression for the potential due to a continuous distribution of matter,

$$\phi(\vec{x}) = -G \int \frac{\rho(r')}{|\vec{x} - \vec{x}'|} d^3x'$$

prove that a spherically symmetric body with an arbitrary radial dependent mass density, $\rho(r')$, behaves like a point mass with total mass $M = \int d^3x' \rho(r')$ concentrated at the center of the sphere for all points, \vec{x} , lying outside of the sphere.

4. (30 min) A particle of mass M is constrained to move on the surface of an infinitely long cylinder of radius R . (The cylinder's axis is the z -axis and its circular cross-section is in the x - y plane). The particle is subject to a force directed toward the origin and proportional to the distance of the particle from the origin: $\vec{F} = -k \vec{r}$ $\vec{r} = x \vec{i} + y \vec{j} + z \vec{k}$.
- (a) Find the Lagrangian and Hamiltonian of the problem.
 - (b) Find the equation of motion.
 - (c) Characterize the motion perpendicular to the z -axis.
 - (d) Characterize the motion parallel to the z -axis.

5. (30 min) A solid sphere of mass M and radius R rotates freely in space with an angular velocity ω about a fixed diameter. A particle of mass m , initially at one pole, moves with a constant velocity v along a great circle of the sphere. Show that, when the particle has reached the other pole, the rotation of the sphere will have been retarded by an angle

$$\alpha = \omega T \left[1 - \left\{ \frac{2M}{2M + 5m} \right\}^{1/2} \right],$$

where T is the total time required for the particle to move from one pole to the other.

6. (30 min) A pendulum consists of a mass m suspended by a massless spring with unextended length b and spring constant k . The pendulum's point of support rises vertically with a constant acceleration a .
- (a) Using the Lagrangian method find the equations of motion.
 - (b) Determine the Hamiltonian and Hamilton's equations of motion.
 - (c) Find the period of small oscillations.

Part II

Quantum Mechanics

Work five (5) of the six problems given below. Be sure to work each problem on a separate page since different persons will be grading the examination. The time to work each problem is given in parenthesis. Be sure to budget your time wisely, and spend no longer than 30 minutes on a problem. Go on to the next problem if you do not complete one within the allocated 30 minutes.

1. (30 min) The Hamiltonian $H_0 + V$ for a particle on a circle is given by

$$H_0 = -\frac{\hbar^2}{2I} \frac{\partial^2}{\partial \phi^2} \quad \text{and} \quad V = \alpha \cos^2 \phi .$$

Here, I and α are constants, ϕ ranges from 0 to 2π , and V is considered to be a perturbation.

- (a) Find the unperturbed eigenvalues and eigenfunctions of H_0 .
 (b) Using perturbation theory, obtain the lowest order corrections to the eigenvalues and eigenfunctions of the ground state and first excited state.

2. (30 min) (a) Assume a Hamiltonian has a normalized ground state eigenket and energy given by:

$$H |\psi_0\rangle = E_0 |\psi_0\rangle, \quad \langle \psi_0 | \psi_0 \rangle = 1.$$

Given that H has a discrete spectrum, show, by inserting a complete set of states, that the normalized expectation value of the Hamiltonian in an arbitrary nonnull ket $|\psi\rangle$,

$$E' \equiv \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle},$$

is always greater than or equal to the ground-state energy:

$$E' \geq E_0.$$

- (b) Show that under the linear (in λ) change in the wavefunction,

$$|\psi\rangle = |\psi_0\rangle + \lambda |\psi_1\rangle,$$

$$\langle \psi_0 | \psi_0 \rangle = 1,$$

$$\langle \psi | \psi \rangle = 1,$$

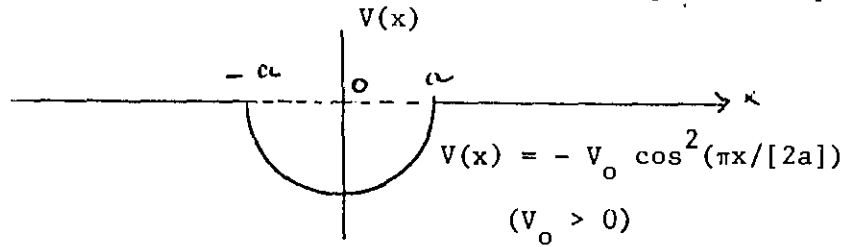
where $|\psi_0\rangle$ is the exact ground state wave function of H , λ is an arbitrary complex number, and $|\psi_1\rangle$ is an additional arbitrary ket, that the associated change in the expectation value of H is quadratic in λ :

$$E' = \langle \psi | H | \psi \rangle = E_0 + \alpha |\lambda|^2.$$

Find the value of α and show that

$$\alpha \geq 0.$$

3. (30 min) Consider the one-dimensional well in a flat potential plane:

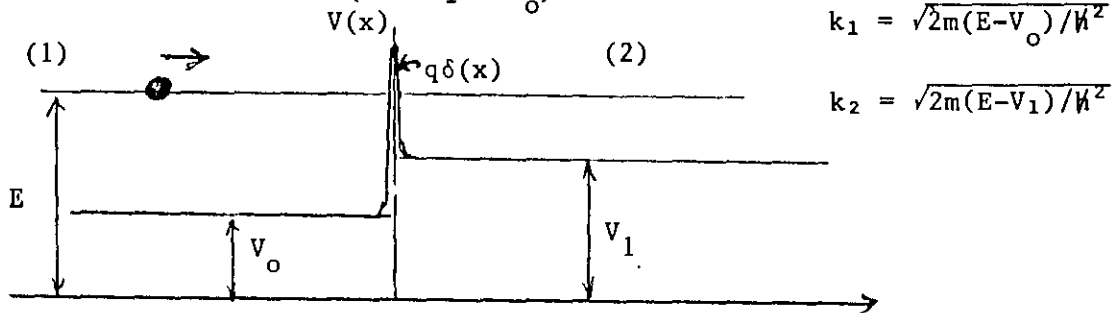


Using the WKB bound-state approximation,

$$\int_{x_1}^{x_2} dx k(x) = \pi (n + \frac{1}{2}), \quad (n = 0, 1, 2, \dots)$$

where $x_{1,2}$ are the classical turning points and $k(x)$ is the position-dependent wave number, estimate the well depth, $(V_0)_{\min}$, such that a single bound state exists for the given potential.

4. (30 min) A δ -function potential $V(x) = q \delta(x)$ [q a constant, $\delta(x) = \infty$ at $x = 0$, $\delta(x) = 0$ for $x \neq 0$, $\int \delta(x) dx = 1$] is superimposed at the origin ($x = 0$) on top of two potential steps $V(x) = V_0$ for $x < 0$ and $V(x) = V_1$ for $x > 0$. A particle of mass m is incident from the left ($E > V_1 > V_0$).



- Write down the wave functions for regions (1) and (2). Express your answers in terms of k_1 and k_2 .
 - Indicate which part of your solution corresponds to a wave traveling to the left and which part to the right. Prove your answer.
 - Notice that the presence of the δ -function causes an adjustment in the boundary conditions. Establish what this adjustment is and obtain the reflection coefficient R and the transmission coefficient T and show that $R + T = 1$.
5. (30 min) A particle of mass m is attached to the end of a rigid rod of negligible mass and of length B . The other end of the rod, located at the origin, is fixed to a bearing such that the rod may rotate in the x - y plane.
- Write down the Schrödinger equation.
 - Solve the equation to obtain expressions for energy and wave function for the system.
 - Calculate the expectation value of the angular momentum of the system.
 - If a series of measurements of angular momentum (in any given state) are made on the system, do you expect always to get the same value (besides the instrument-related non-reproducibility)? Justify your answer.

6. (30 min) Measurement of the z-component of the spin of a neutron reveals the value $S_z = \hbar/2$. What spin state is the particle in after the measurement? Show (by direct calculation) that in this state

$$\langle S_x^2 \rangle = \langle S_y^2 \rangle = \frac{1}{2} \langle S^2 - S_z^2 \rangle = \hbar^2/4 = \langle S_z^2 \rangle.$$

- (b) Now consider a polarized beam containing electrons in the α_z state which has been sent through a Stern-Gerlach analyzer which measures S_x . What values will be found and with what probabilities will these values occur?

Part III

Electricity and Magnetism

Work five (5) of the six problems given below. Be sure to work each problem on a separate page since different persons will be grading the examination. The time to work each problem is given in parenthesis. Be sure that you do not exceed this time on any given problem. Go on to the next problem after the time is up even though you may not have completed the problem.

1. (30 min) A long iron pipe (inner radius a , outer radius b , permeability μ) is placed at right angles to the field in a region of initially uniform magnetic induction \vec{B}_0 . Show that the field inside the pipe is

$$\frac{4 \mu \vec{B}_0}{(\mu + 1)^2 - (\mu - 1)^2 a^2/b^2}$$

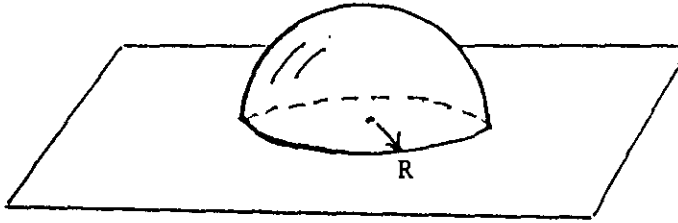
2. (30 min) Let n be the density of electrons of mass m , charge e , and velocity \vec{v} in a semi-infinite perfect conductor. What is the time derivative of the magnetic induction \vec{B} inside the conductor? To answer this question, follow steps (a) and (b) below.
- (a) First, derive a relation between the time derivative of the current density $\partial \vec{J} / \partial t$ and $\partial \vec{B} / \partial t$ when an electric field \vec{E} is applied.
- (b) Next, show that $\partial \vec{B} / \partial t$ is very small except near the surface of the semi-infinite perfect conductor. If n is one electron per atom, to what depth does $\partial \vec{B} / \partial t$ penetrate into the conductor? [note the identity: $\nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$].
- (c) It is known experimentally that superconductors have $\vec{B} \cong 0$ inside (the Meissner Effect) rather than $\partial \vec{B} / \partial t \cong 0$. What change in the relation in (a) would you propose to account for this? How would \vec{B} then vary in the superconductor? (The altered equation is, of course, the London equation).
3. (30 min) (a) Prove Green's reciprocity theorem: If ϕ_1 is the potential due to a volume charge density, ρ_1 , within a volume V and a surface charge density, σ_1 , on the conducting surface bounding V , while ϕ_2 is the potential due to another charge distribution ρ_2 and σ_2 , then

$$\int_V \rho_1 \phi_2 d^3x + \int_S \sigma_1 \phi_2 da = \int_V \rho_2 \phi_1 d^3x + \int_S \sigma_2 \phi_1 da$$

[Hint: This theorem compares two separate problems with, however, the same geometry. Try evaluating $\int d^3x \vec{E}_1 \cdot \vec{E}_2$ two ways].

- (b) Apply this theorem to two concentric circular conducting cylinders of radii a and b ($b > a$) with a point charge, q , placed between them. Find the induced charges on the inner (Q_a) and outer (Q_b) surfaces. [Hint: Take as a comparison problem the two cylinders at unequal potentials with no point charges present].

4. (30 min)



A hemispherical bump of radius R rises from an infinite two-dimensional plane. All surfaces are conducting. Using a set of image charges find the Dirichlet Green function, $G(\vec{x}, \vec{x}')$, at all points above the conducting surface.

5. (30 minutes) Suppose that you have an electric charge e and a magnetic charge g separated by a distance d . The field of the electric charge is

$$\vec{E} = \frac{e}{4\pi\epsilon_0} \frac{\vec{r}}{r^3}$$

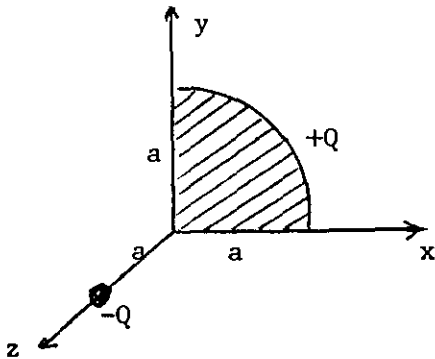
of course, and the field of the magnetic charge is

$$\vec{B} = \frac{\mu_0 g}{4\pi} \frac{\vec{r}}{r^3}$$

Find the total angular momentum stored in the resulting electromagnetic fields. (This system is known as Thomson's monopole.) Show that your answer is independent of the separation distance and points in the direction from e toward g . What does this suggest about the electric and magnetic charge if it can be shown that magnetic monopoles exist? (This idea was first proposed in 1931 by Dirac.)

(30 min)

6. / Consider a flat uniform rigid body of charge Q in the shape of one quarter of a circle of radius a and lying between the x and y axes as shown. There exists a negative point charge $-Q$ on the z -axis at distance a from the xy -plane as shown.



(a) Find the dipole moments p_x and p_z for the distribution.

(b) Find the quadrupole moments Q_{xx} and Q_{xy} .

(Express all answers in both parts in terms of Q and a).

Work five (5) of the six problems given below. Be sure to work each problem on a separate page since different persons will be grading the examination. The time to work each problem is given in parenthesis. Please do not spend more than 30 minutes on any give problem. Go on to the next problem even though you may not have completed the one you are working on.

1. (30 minutes) A particle i has coordinate $\mathbf{r}_i = (q_{ix}, q_{iy}, q_{iz})$ when the force acting on it is $\mathbf{F}_i = dp_i/dt$, where $\mathbf{p}_i = (p_{ix}, p_{iy}, p_{iz})$. The virial of a system of n particles is given as

$$C \equiv -\frac{1}{2} \sum_{i=1}^n \overline{\mathbf{F}_i \cdot \mathbf{r}_i}$$

where the bar denotes a time average.

a. Assuming (i) the Hamiltonian equations of motion and (ii) the ergodic hypothesis that ensemble and time averages yield identical results, prove that

$$C = \frac{3}{2} nkT$$

b. If the forces are derivable from a potential W ,

$$F_{ij} = -\frac{\partial W}{\partial q_{ij}}$$

and the momenta are involved only in a kinetic energy of the form

$$K = \sum_{i=1}^n \frac{p_i^2}{2m}$$

prove that

$$\overline{K} = \frac{1}{2} \sum_i \overline{\nabla W \cdot \mathbf{r}_i} = \frac{3}{2} nkT$$

c. For a single particle ($n = 1$) under a central force of potential energy

$$W = ar^u$$

prove that

$$\overline{K} = \frac{u}{2} \overline{W} = \frac{u}{2+u} \overline{E}$$

where E denotes the total energy.

2. (30 minutes) Two fluids F_1, F_2 of fixed volumes and constant heat capacities C_1, C_2 are initially at temperatures T_1, T_2 ($T_1 > T_2$), respectively. They are adiabatically insulated from each other. A quasistatically acting Carnot engine E uses F_1 as heat source and F_2 as heat sink, and acts between the systems until they reach a common temperature, T_0 .

- a. Obtain an expression for T_0 and for the work done by the Carnot engine.
- b. If a common temperature is established by allowing direct heat flow between F_1 and F_2 , what is the final temperature and what is the change in entropy?
- c. Show that for all positive C_1, C_2 this change is an increase. (Does this make sense?)

3. (30 min) Consider a string of length L with fixed ends. Write down the equation of motion for small amplitude vibrations. Next, solve for the normal modes of vibration and give the general solution for an arbitrary initial state. Finally, compute the Green function $G(x, x'; k)$ for the operator $d^2/dx^2 + k^2$ by solving the equation:

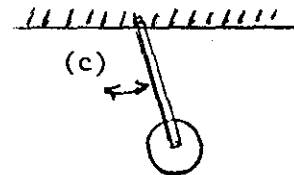
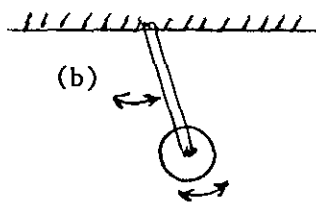
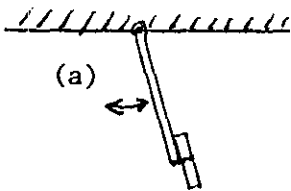
$$\left(\frac{d^2}{dx^2} + k^2 \right) G(x, x'; k) = \delta(x - x')$$

subject to the boundary condition that the string is fixed at both ends.

4. (30 min) A uniform rod has mass m and length L . A uniform disk of mass M and radius R is attached to the end of the rod by means of an axle which is perpendicular to both the rod and the disk. The rod is suspended at the other end and caused to oscillate through a small angle as a pendulum. Consider the three cases:

- (a) The disk is at a right angle to the plane of oscillation.
- (b) The disk is in the plane of oscillation and free to rotate about its axis.
- (c) The disk is in the plane of oscillation but fixed relative to the rod.

For each of the three cases, calculate the period for small oscillations.



5. (30 min) Consider a gas of N identical spin $\frac{1}{2}$ fermions of nonrelativistic energy

$$\epsilon_p = \vec{p}^2/2m,$$

in an enclosure of volume V . These particles are placed in a magnetic field of strength B pointing in the z -direction, which changes their energies to

$$\epsilon_{p,B} = \vec{p}^2/2m + \lambda B \sigma_z', \quad (\sigma_z = \pm 1).$$

Call $\sigma_z' = 1$ "spin up" and $\sigma_z' = -1$ "spin down."

- (a) At zero temperature, give expressions for the chemical potentials, μ_{\pm} , of the spin up and down fermions in terms of N_{\pm} (the number of spin up or down particles), V , m , and λB . [Hint: N_{\pm} are given by

$$N_{\pm} = \frac{8\pi V}{h^3} \int_0^{p_F^{\pm}} dp p^2, \quad (T = 0)$$

where p_F^{\pm} are the Fermi momenta of the spin up or down particles].

- (b) What relation must exist between μ_+ and μ_- in thermal equilibrium? (This is assuming that spin up can change to spin down and vice versa through collisions).
(c) Using the relation in (b) and assuming that $N_+ - N_- \ll N$, where $N_+ + N_- = N$ (N a constant), show that the magnetization,

$$m \equiv \frac{\lambda}{V} (N_+ - N_-),$$

is given at $T = 0$ by

$$m = \frac{3\lambda^2 B}{2kT_F} (N/V),$$

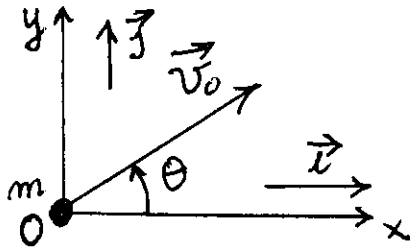
where the "Fermi temperature", T_F , is given by $kT_F = \frac{1}{2m} (3Nh^3/[16\pi V])^{2/3}$.

- 6 (30 min) (a) Find the relative numbers of hydrogen atoms that are in the ground state and in the first, second, and third excited states ($n = 2, 3, 4$) in the solar chromosphere. Assume T to be 5000 K and remember to include the statistical weights of the various levels. [Hint: Use $\epsilon_n = -13.6/n^2$ eV, remember that the degeneracy of each level is $2n^2$ since each level can have two electrons of different spin, and use $k = 8.616 \times 10^{-5}$ eV/Kelvin. You will need a hand calculator to make the calculations].

- (b) Why does the Balmer series appear prominently in absorption in the Solar spectrum?

Work five (5) of the six problems given below. Be sure to work each problem on a separate piece of paper since different persons will be grading the examination. The time to work each problem is given in parenthesis. Distribute your time wisely. Spend no more than 30 minutes on each problem. If you haven't finished a problem, then go on to the next one.

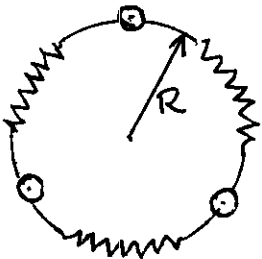
1. (30 min.)



A projectile of mass m is fired with initial velocity \vec{v}_0 from the origin of an xy -coordinate system in a vertical plane as shown to the left. The velocity vector \vec{v}_0 makes an angle θ with the x -axis. The projectile is influenced by a gravitational force $\vec{F}_g = -mg\vec{j}$, where \vec{j} is a unit vector in the positive y -direction. The projectile also encounters a resistive force $\vec{F}_{res} = -\beta\vec{v}$, where β is a positive constant and $\vec{v} = v_x\vec{i} + v_y\vec{j}$ (\vec{i}, \vec{j} are unit vectors)

- Find the vector velocity \vec{v} as a function of time so that at $t = 0$, $\vec{v}(0) = v_0\cos\theta\vec{i} + v_0\sin\theta\vec{j}$. Hint: It may be easier to work out the x - and y -components independently.
- Find the position vector $\vec{r} = x\vec{i} + y\vec{j}$ as a function of time so that $x = 0$, $y = 0$, at $t = 0$.
- Find the time for the projectile to reach its highest point.
- Find the maximum height reached by the projectile.

2. (30 min)



Three beads each of mass M slide without friction on a circular wire of radius R . Each pair of beads is connected by a spring. The three springs are identical. Each has the same spring constant (K) and is unstretched when the beads are equally spaced. Solve for the normal modes of oscillation, and give the frequencies of each mode. Sketch the configuration and motion in each of the three modes.

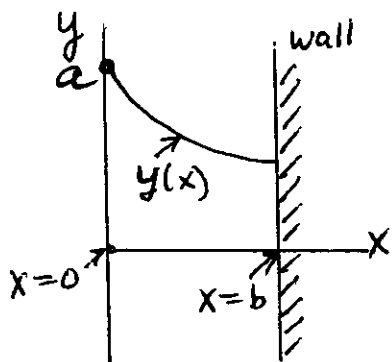
- (30 min) A mass M moves horizontally along a smooth rail. A pendulum is hung from M with a weightless rod and mass m at its end. Find the eigenfrequencies and describe the normal modes.

(continued next page)

4. (30 min) Consider a space craft mission to Venus or Mars. Using basic gravitational force and energy considerations, and no interaction between the planets and the craft following launch from Earth orbit, find the following:
- Orbital velocity of Earth, in a circular orbit.
 - Orbital velocity of the space craft at altitude of 400 km above Earth's surface.
 - For a lowest energy consideration, consider the launch from Earth orbit to be opposition from the sun and to be at aphelion or perihelion in solar orbit. Find the aphelion velocity needed for a perihelion rendezvous with Venus. Then find the change in velocity needed for launch into heliocentric orbit from Earth orbit.
 - Find the travel time for the trip to Venus.
 - Find the perihelion velocity needed for an aphelion rendezvous with Mars. Then find the change in velocity needed for launch into heliocentric orbit from Earth orbit.
 - Find the travel time for the trip to Mars.

Data: $R_{S-E} = 149.6 \times 10^6$ km $M_E = 5.98 \times 10^{24}$ kg
 $R_{S-V} = 108.2 \times 10^6$ km $M_S = 1.99 \times 10^{30}$ kg
 $R_{S-M} = 228.0 \times 10^6$ km $G = 6.67 \times 10^{-11}$ N-m²/kg²
 $R_E = 6378$ km

5. (30 min) (Calculus of variations) Find the equation for the surface of revolution



(revolve the $y(x)$ line shown about the x -axis) which connects the point at $x = 0, y = a$ to the fixed wall at $x = b, y = \text{unknown}$ ($b \ll a$) and which has the minimum surface area, A . [Hints: In addition to the usual Euler-Lagrange equation, there is a subsidiary endpoint condition at $x = b$. The functional form turns out to be

$$y(x) = c \cosh ([x-d]/c),$$

where c, d are constants somehow related to a, b above. Note that the condition $b \ll a$ removes the possibility of certain discontinuous minimums.]

6. (30 min) (i) A layer of dust is formed h feet thick (h small compared to the earth's radius) by the fall of meteors reaching the earth from all directions. By considering the angular momentum, find the change in the length of the day, in terms of R (radius of the earth), h , d (density of the dust) and D (density of the earth). Use a notation in which the initial quantities carry subscript zero and final quantities a subscript of 1. The moment of inertia of a solid sphere about an axis through its center is $(2/5)MR^2$ and that of a thin-walled, hollow sphere of mass m , radius R , is $(2/3)MR^2$.

(ii) If the solar system were immersed in a uniformly dense spherical cloud of weakly-interactive massive particles (WIMPs), then objects in the solar system would experience gravitational forces from both the sun and cloud of WIMPs such that

$$F_r = -\frac{k}{r^2} - br.$$

Assume that the extra force due to the WIMPs is very small (i.e., $b \ll k/r^3$). Find the frequency of radial oscillations for a nearly circular orbit and the rate of precession of the perihelion of this orbit.

Quantum Mechanics

Work five (5) of the six problems given below. Be sure to work each problem on a separate page since different persons will be grading the examination. The time to work each problem is given in parenthesis. Be sure to budget your time wisely, and spend no longer than 30 minutes on a 30-minute problem or no longer than 35 minutes on a 35-minute problem. Go on to the next problem if you do not complete a problem within the allotted time.

1. (35 min) You are given the three Cartesian/^{spin} components of a particle of spin 1/2 as $S_x = (\hbar/2)\sigma_x$, $S_y = (\hbar/2)\sigma_y$, $S_z = (\hbar/2)\sigma_z$, where σ_i are the 2x2 Pauli spin matrices.
- Find the normalized eigenvectors for S_x .
 - Show that the matrix \mathcal{U} having as columns the normalized eigenvectors for S_x is unitary and that it transforms S_x into diagonal form. Write down the expression for the diagonal form.
 - Show that \mathcal{U} can be constructed from the rotation operator $\mathcal{U} = \exp(\pm i\pi S_y / (2\hbar))$. Which sign must be used?
 - What is the meaning of the transformation from the point of view of a rotation in 3-D space? Is the rotation proper or improper?
 - If a particle has a spin of $\hbar/2$ and a measurement is made of the sum of S_x and S_z , what values would be found?

2. (30 min) (a) Write down the non-relativistic quantum mechanical differential equation which describes the hydrogen atom. State explicitly what the variables in the equation mean.

- (b) For the stationary bound state problem, separate the equation into radial and angular parts.

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \phi^2}$$

- (c) Employing the appropriate boundary conditions, analyze the radial equation using the Sturm-Liouville technique. What conclusions can you draw about the solutions?

- (d) Analyze the radial equation. First describe the singularity structure of the equation in the complex plane. Using an appropriate series, find the solution of the radial equation about $r = 0$.

3. (30 min) (a) Starting with the eigenvalue-eigenvector equation (λ is some parameter in the theory)

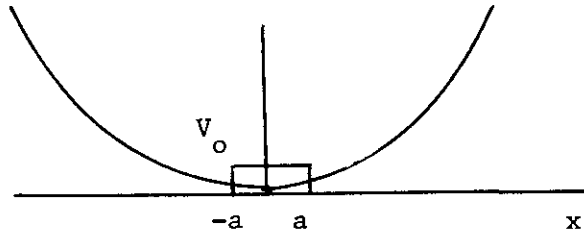
$$H(\lambda) |\psi(\lambda)\rangle = E(\lambda) |\psi(\lambda)\rangle,$$

where $H(\lambda)$ is the Hamiltonian, $E(\lambda)$ is the energy and $|\psi(\lambda)\rangle$ is the normalized eigenstate ($\langle \psi(\lambda) | \psi(\lambda) \rangle = 1$), show that

$$\langle \psi(\lambda) | \frac{\partial H}{\partial \lambda} | \psi(\lambda) \rangle = \frac{\partial E(\lambda)}{\partial \lambda}.$$

(problem continued on next page)

3. (continued) (b) A harmonic oscillator potential has a small bump at $x = 0$ as shown:



The one-dimensional Hamiltonian is $H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 + V(x)$,

$$V(x) = V_0, \quad |x| \leq a$$

0, elsewhere.

The constant a^2 is quite small ($a^2 \ll \hbar/(m\omega)$) and the ground state unperturbed energy and wavefunction are

$$E_0 = \hbar \omega / 2,$$

$$u_0(x) = (m\omega/(\pi\hbar))^{1/4} \exp(-[m\omega^2 x^2]/[2\hbar]).$$

By using perturbation theory and part (a) above, find the first order effect of the potential $V(x)$ on the expectation value of x^2 in the ground state:

$$\langle x^2 \rangle = \hbar/(2m\omega) + ?$$

(a)

4. (35 min) Show that the Lippmann-Schwinger equation for outgoing waves $|\psi^{(+)}\rangle$,

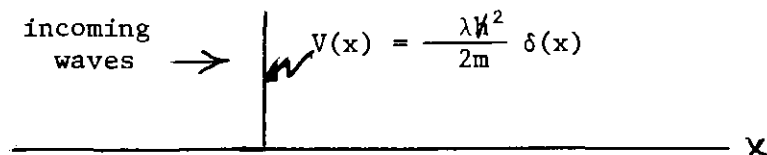
$$|\psi^{(+)}\rangle = |\phi\rangle + \frac{1}{E - H_0 + j\epsilon_0} V |\psi^{(+)}\rangle,$$

($E = \text{energy} = k^2 \hbar^2 / (2m)$, $H_0 = p^2 / 2m$ where p is the momentum operator, and

$|\phi\rangle$ is the H_0 energy eigenket) in one dimension is equivalent to the integral equation:

$$\langle x | \psi^{(+)} \rangle = \langle x | \phi \rangle - \frac{i\pi}{k\hbar^2} \int_{-\infty}^{+\infty} dx' e^{ik|x-x'|} V(x') \langle x' | \psi^{(+)} \rangle.$$

(b) A delta-function potential is located at the origin as shown:



Use part (a) to find the transmission coefficient ($x > 0$), t , assuming

$$\langle x | \phi \rangle = \frac{e^{ikx}}{(2\pi\hbar)^{1/2}}$$

$$t \equiv \frac{\langle x | \psi^{(+)} \rangle}{\langle x | \phi \rangle} \quad \text{for this potential.}$$

5. (30 min) A particle of mass m interacts in three dimensions with a spherically symmetric potential of the form

$$V(\vec{r}) = -c \delta(|\vec{r}| - a).$$

In other words, the potential is a delta function that vanishes unless the particle is precisely a distance "a" from the center of the potential. Here c is a positive constant.

- (i) Find the minimum value of c for which there is a bound state.
 (ii) Consider a scattering experiment in which the particle is incident on the potential with a low velocity. In the limit of small incident velocity, what is the scattering cross-section? What is the angular distribution?

6. (30 min) (a) In time-independent perturbation theory, the Hamiltonian can be written as $H = H_0 + H'$ (H' is the perturbation).

$$\text{Show that } \sum_{nm} |H'_{nm}|^2 = (H'^2)_{nn}.$$

- (b) Consider a hydrogen atom in its ground state in the presence of a uniform electric field E , where $H' = -eEz$. Show that the first-order correction to the ground state energy H'_{11} is zero.

$$\text{Use } \psi_{100} = \frac{1}{\pi^{1/2} a_0^{3/2}} e^{-r/a_0}, \text{ where } a_0 = \text{Bohr radius.}$$

- (c) Show that the matrix element H'_{1q} is non-zero only for $q = 1$.

$$Y_{\ell m} = \left[\frac{2\ell+1}{4\pi} \frac{(\ell - |m|)!}{(\ell + |m|)!} \right]^{1/2} P_{\ell}^m(\cos\theta) e^{im\phi}$$

$$\text{and } \int_{-1}^{+1} P_n^m(\mu) P_{\ell}^m(\mu) d\mu = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} \delta_{n\ell}$$

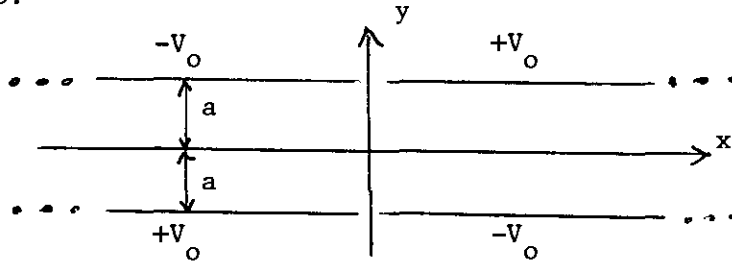
- (d) The change in ground-state energy is $\Delta W = \frac{1}{2} \alpha E^2$, where $\alpha =$ polarizability and is about $0.68 \times 10^{-24} \text{ cm}^3$. Simple estimates indicate that $|H'_{1q}|^2$ decreases rapidly as the quantum number n associated with state q increases:

$$\frac{1}{2} \alpha E^2 = \sum_{q \neq 1} \frac{|H'_{1q}|^2}{E_q - E_1}.$$

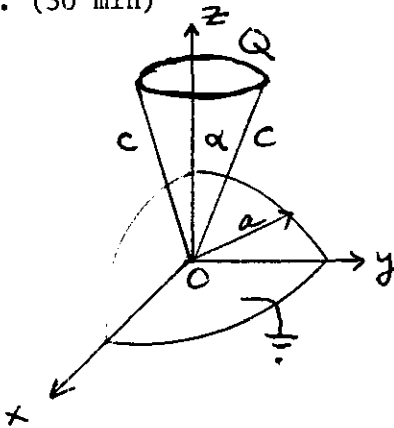
Show that $\frac{1}{2} \alpha E^2 \propto \frac{(H'^2)_{11}}{E_2 - E_1}$, where E_2 is the lowest P-state energy.

Work five (5) of the six problems given below. Be sure to work each problem on a separate page since different persons will be grading the examination. The time to work each problem is given in parenthesis. Be sure that you do not exceed this time on any given problem. Go on to the next problem after the time is up even though you may not have completed the problem.

1. (30 min) Four large conducting plates are arranged as in the figure below, and are maintained with the electrostatic potentials as shown. The plate separation of $2a$ is very much smaller than the spatial extent of the plates in the x - and z -directions. Find the electrostatic potential $\phi(x,y,z)$ in the region between the plates.



2. (30 min)



A grounded conducting spherical shell of radius a has its center on the z -axis of a charged circular ring (of charge Q) as shown in the figure to the left. The center of the sphere is at the origin of a rectangular coordinate system, and the charged circular ring is in a plane parallel to the xy plane. The cone subtended by the ring has half-angle α with the z -axis. The distance from O to any point on the ring is c .

Show that the force that pulls the sphere into the ring is

$$\frac{-Q^2}{4\pi\epsilon_0 c^2} \sum_{n=0}^{\infty} (n+1) P_{n+1}(\cos\alpha) P_n(\cos\alpha) \left(\frac{a}{c}\right)^{n+1}$$

(SI units are employed)

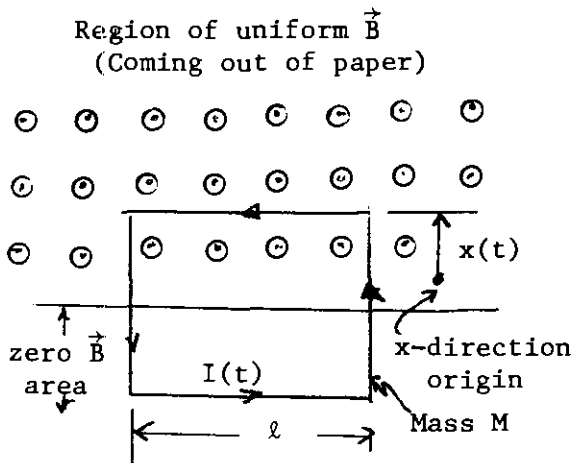
HINT: The potential for the charged ring for $r < c$, $\theta \neq \alpha$, is

$$V = \frac{Q}{4\pi\epsilon_0 c} \sum_{n=0}^{\infty} (r/c)^n P_n(\cos\alpha) P_n(\cos\theta).$$

The ring charge induces a charge in the spherical shell. Determine the form of the electrostatic potential for the sphere, use suitable boundary conditions, calculate E_z , and then obtain F_z .

$$P_n' = \frac{n(n+1)}{2n+1} \left[\frac{P_{n-1} - P_{n+1}}{1 - \cos^2\alpha} \right].$$

3. (30 min)



A very long flat circuit of mass M and self-inductance L is superconducting so that its resistance is zero. The end of the circuit is immersed in a uniform magnetic field \vec{B} directed out of the page.

Find the motion of the circuit as a function of time. You may assume that

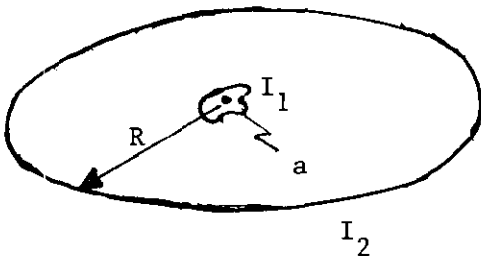
$$\dot{x}(0) = x(0) = 0,$$

and that the initial non-zero current in the circuit is $I(0)$. (The motion is in a horizontal plane so gravity is not a factor and the circuit moves frictionlessly).

4. (30 min) A particle of mass m and charge e moves at constant, nonrelativistic speed v_1 in a circle of radius a .

- What is the power emitted per unit solid angle in a direction at angle θ to the axis of the circle?
- Now suppose a particle is moving nonrelativistically in a constant magnetic field B . Find the frequency of circular motion and the total emitted power. Show that the total emitted power is emitted solely at the frequency ω_B .
- A pulsar is believed to be a rotating neutron star. Such a star is likely to have a strong magnetic field, B_0 , since it traps lines of force during its collapse. If the magnetic axis of the neutron star does not line up with the rotation axis, there will be magnetic dipole radiation from the time-changing magnetic dipole, $\vec{m}(t)$. Assume that the mass and radius of the neutron star are M and R , respectively; that the angle between the magnetic and rotation axis is α ; and that the rotational angular velocity is ω . Find an expression for the radiated power P in terms of ω , R , B_0 , and α . Assuming that the rotational energy of the pulsar is the ultimate source of the radiated power, find an expression for the slow-down time scale, τ , of the pulsar.

5. (20 min)



A small planar circuit of arbitrary shape and area a is located at the center of a much larger circular circuit of radius R . The planes of the two circuits are coincident. Find the mutual inductance of this system.

(continued next page)

6. (30 min) A long dielectric cylinder of radius a and dielectric constant κ is placed in an initially uniform electric field \vec{E}_0 . The axis of the cylinder is oriented at right angles to the direction of \vec{E}_0 . The cylinder contains no external charge. Determine the electric field at points inside and outside the cylinder.

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

Work five (5) of the six problems given below. Be sure to work each problem on a separate page since different persons will be grading the examination. The time to work each problem is given in parenthesis. Please do not spend more than 30 minutes on any given problem. Go on to the next problem even though you may not have completed the one you are working on.

(a)

1. (30 min)/Consider a classical one-dimensional system of noninteracting diatomic molecules enclosed in a box of length L at temperature T . The Hamiltonian for a single molecule is (center of mass system)

$$H = \frac{p^2}{2\mu} + \frac{1}{2} \mu \omega^2 x^2,$$

where $p = \mu \dot{x}$ and x is the relative distance coordinate (μ is the reduced mass).

Find the average intermolecular distance squared, $\langle x^2 \rangle$.

- (b) Now consider the same Hamiltonian in quantum mechanics. The energies of the molecules are given by

$$E_n = \hbar \omega (n + \frac{1}{2}), \quad n = 0, 1, 2, \dots$$

Again, find the average intermolecular distance squared at temperature T . Show that the classical result in part (a) is recovered as $T \rightarrow \infty$. [Hint: In terms of raising and lowering operators A^+ , A ,

$$\begin{aligned} A^+ |n\rangle &= \sqrt{n+1} |n+1\rangle \\ A |n\rangle &= \sqrt{n} |n-1\rangle, \end{aligned}$$

one has $x = \sqrt{\hbar/(2m\omega)} (A + A^+)$.

A.

2. (30 min)/Using (1) the mean kinetic energy per molecule of a gas at temperature T , and (2) the expression for the height of a particle with initial velocity v (surface of earth) will rise before falling back to the earth:

- (a) Derive an equation for the characteristic height of a molecule of mass m and temperature T .
 (b) At $T = 250$ K, compute the characteristic heights for N_2 , O_2 , CO_2 , and H_2 .
 (c) What does this indicate about the compositional structure of the earth's atmosphere? Explain.

B. Use the strength of the magnetic field at the earth's surface to:

- (a) Estimate the strength in the two radiation zones (Van Allen belts).

Use $B = 0.4 \times 10^{-4}$ T @ $r_E = 1$; r for inner zone = $1.5 r_E$;

r for outer zone = $3.5 r_E$; and magnetic field decreasing away from earth as $1/r_E$.

- (b) Then, estimate the radius of curvature of a 50 MeV proton in the two zones.

3. (30 min) A string vibrates with fixed ends at $x = 0$ and $x = L$. Write down the equation of motion for small amplitude vibrations. Solve for the characteristic (normal) modes of the system and give an expression for the general solution of the equation of motion. Finally, compute the Greenfunction $G(x, x'; k)$ for the operator $d^2/dx^2 + k^2$ ($0 \leq x \leq L$) by solving the equation:

$$(d^2/dx^2 + k^2) G(x, x'; k) = \delta(x - x')$$

subject to the same boundary conditions as possessed by the vibrating string.

4. (30 min)

- A. Apply the Boltzmann equation to a neutral hydrogen atom and derive an expression for the population of the n^{th} energy level relative to that of the ground state at temperature T . Assume that the multiplicity of each level is unity, construct an approximate graph showing results for $T \times 6000\text{K}$.
- B. Let N_2 be the number of 2nd-level (1st excited level) hydrogen atoms and ground state N_1 ; use the figure below to find the excitation ratio (N_2/N_1) and the excited fraction (N_2/N) for each of the following stars:
- (a) Sirius, $T = 10,000\text{K}$
 - (b) Rigel, $T = 15,000\text{K}$
 - (c) Sun, $T = 5,700\text{K}$

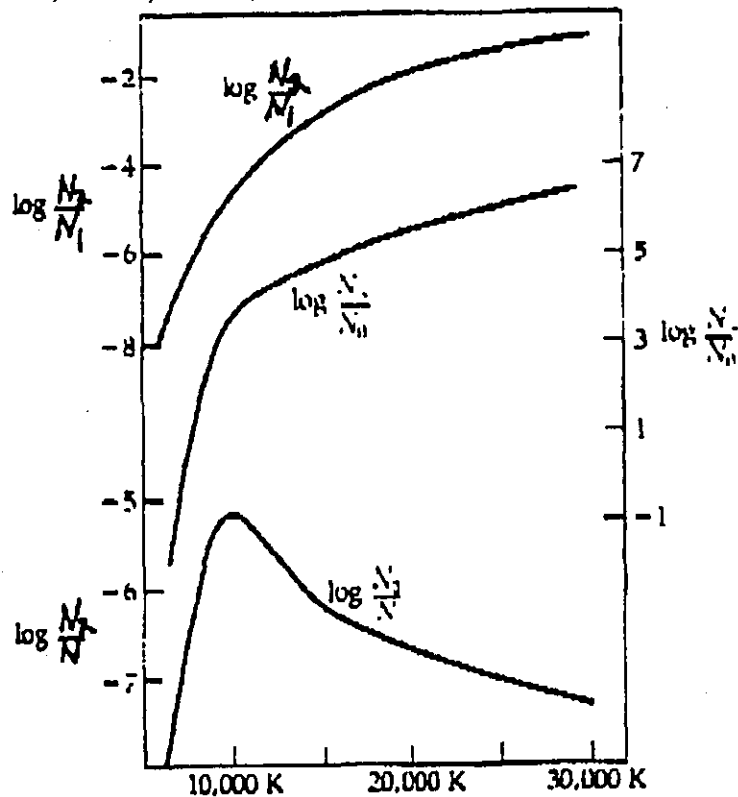


Figure B-13

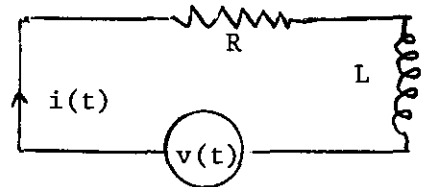
Excitation and ionization curves for hydrogen Balmer lines. The relative populations of the energy levels (N_2/N_1) from the Boltzmann equation and the ionization stages (N_+/N_0) from the Saha equation are calculated for equilibrium at the indicated temperatures. The lower curve shows the combination of the upper two with $N = N_0 + N_+$.

- C. Which star will exhibit the strongest Balmer absorption lines? Explain the reasoning in arriving at your answer.

5. (30 min) A monatomic Boltzmann ideal gas of spin $1/2$ atoms in a uniform magnetic field has, in addition to its usual kinetic energy, a magnetic energy of $\pm mB$ per atom, where m is the magnetic moment. (It is assumed that the gas is so dilute that the interaction of magnetic moments may be neglected.)

- (i) What is the partition function for a canonical ensemble of N such atoms?
- (ii) Find the energy.
- (iii) Calculate C_v from the partition function.

6. (30 min) For the RL circuit shown, do the following:



- (a) Write down the equation for the current $i(t)$ corresponding to an arbitrary applied voltage $v(t)$.
- (b) Using Laplace transform methods, find the Laplace transform of the current $I(s)$ in terms of the voltage transform

$$V(s) = \int_0^{\infty} e^{-st} v(t) ds$$

- (c) Compute $V(s)$ for the specific voltage

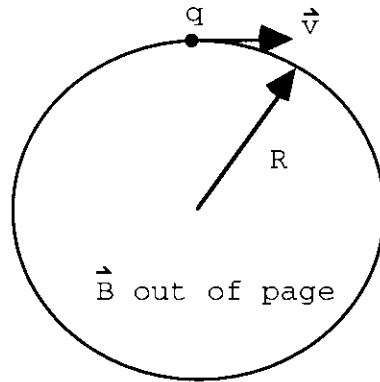
$$v(t) = V_0 \sin \omega t, \quad t > 0$$
$$v(t) = 0, \quad t < 0$$

- (d) For the applied voltage in part (c), compute $i(t)$ by inverting the Laplace transform $I(s)$.

Part I: Classical Mechanics

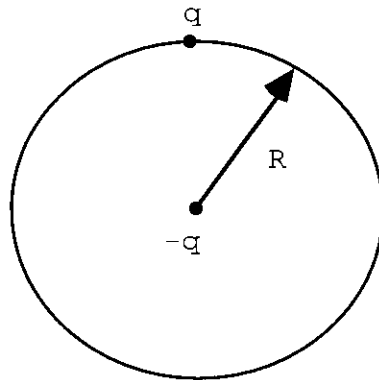
Instructions: Work five (5) of the following six problems. Problems are to be worked on separate sheets of paper to facilitate grading. Each problem should take approximately 30 minutes to solve, with the exception of number six. Don't spend too much of your time on any one problem. Distribute your time wisely.

1. a. A macroscopic charged relativistic particle of mass m travels in a circular path of radius R in a uniform \vec{B} field as shown (the charge q is positive):



Given that the particle's motion has a radius R , find β^2 as a function of R . ($\beta^2 = \frac{v^2}{c^2}$)

- b. Now consider a relativistic charge bound in a circular orbit in the Coulomb field of a much heavier particle with charge $-q$, as shown:



Again, find β^2 as a function of R . [Note: in both parts \vec{v} is measured in a frame of reference stationary with respect to the center of the circles.]

Part I: Classical Mechanics p. 2

2. An ant (of mass m) runs counterclockwise (at speed v_0 relative to the floor) along the rim of a turntable (a disc of radius R and mass M) which rotates clockwise at an angular speed ω_0 (also relative to the floor). The ant then finds a crumb on the rim and stops to eat it.

- a. What is the angular speed after the ant has "stopped" ?
- b. How much energy was involved in the process of the ant stopping ?

3. A thin hoop of radius R and mass M oscillates in its own plane with one point of the hoop fixed. Attached to the hoop is a small mass M constrained to move (in a frictionless manner) along the hoop. Consider only small oscillations and show that the characteristic frequencies are

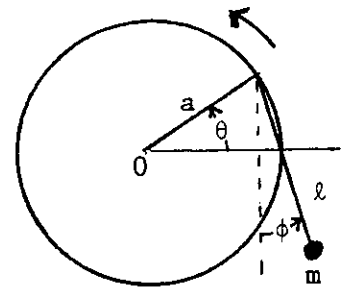
$$\omega_1 = (2g/R)^{1/2}, \quad \omega_2 = (g/2R)^{1/2} .$$

Find the normal modes and describe the physical situation for each mode.

4. a. A homogeneous disk of radius R and mass M rolls without slipping on a horizontal surface and is attracted to a point a distance d below the plane. If the force of attraction is proportional to the distance from the disk's center of mass to the force center, find the frequency of small oscillations around the position of equilibrium.

b. Now consider a homogeneous slab of thickness a , placed atop a fixed cylinder of radius R whose axis is horizontal. Find the condition for stable equilibrium of the slab (assuming no slipping) and the frequency of small oscillations.

5. A simple pendulum of length ℓ (mass of rod = 0), with mass point m has a radius of suspension that rotates with angular velocity $\dot{\theta}$ on the vertical circle of radius a as shown in the figure. All the motion is confined to a single, fixed vertical plane.



- a. Set up the Lagrangian in terms of $\theta, \phi, \dot{\theta}, \dot{\phi}$. (Use 0 as the reference point for the potential energy.)
- b. Use the matrix method to find the Hamiltonian in term of $p_\theta, p_\phi, \theta, \phi$.
- c. Write down Hamilton's four equations using the Hamiltonian from part b.

6. (20 minutes) A particle of mass M in periodic motion in one dimension under the influence of a potential $V(x) = F |x|$, where F is a constant.

- a. Use action-angle variables to find the frequency ν of oscillation of the motion as a function of the particle's energy E .
- b. Check your answer by direct calculation by starting from the conservation of energy equation.

Part II: Quantum Mechanics

Instructions: Work five (5) of the following six problems. Problems are to be worked on separate sheets of paper to facilitate grading. Each problem should take approximately 30 minutes to solve. Don't spend too much of your time on any one problem. Distribute your time wisely.

1. An atom of tritium is in its ground state when the nucleus suddenly decays into a helium nucleus, with the emission of a fast electron which leaves the atom without perturbing the extra-nuclear electron. Find the probability that the resulting He^+ ion will be left in

- a 1s state
- a 2s state
- What is the selection rule for the ℓ quantum number in the transition?

2. a. Given the normalizable one-dimensional wave at $t = 0$,

$$u(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk A(k)e^{ikx},$$

and defining the expectation value in k-space,

$$\langle F(k) \rangle = \int_{-\infty}^{\infty} dk A^*(k)F(k)A(k),$$

show that one may write

$$\langle F(k) \rangle = \int_{-\infty}^{\infty} dx u^*(x,0)F\left(-i\frac{d}{dx}\right)u(x,0).$$

(Consider $F(k)$ a power series in k .)

b. Using part a. (or any other means), now establish that

$$\frac{\hbar\langle k \rangle}{m} = \int_{-\infty}^{\infty} dx j(x,0),$$

where

$$j(x,0) = \frac{i\hbar}{2m} \left(\frac{\partial u^*}{\partial x} u - u^* \frac{\partial u}{\partial x} \right)$$

is the probability current.

3. A spin one particle is in the state $|s = 1, m = 1\rangle$ where $\hat{S}_z|s = 1, m = 1\rangle = \hbar|s = 1, m = 1\rangle$. This particle is then rotated about the y-axis by $\pi/2$. What is the probability that the measurement of \hat{S}_y (on this rotated state) will yield the outcome \hbar ?

4. Consider a harmonic oscillator. The Hamiltonian can be written as $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)$. A "coherent state", $|\alpha\rangle$, is defined as one which satisfies $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, where α is a complex number. Prove that $e^{(\alpha\hat{a}^\dagger - \alpha^*\hat{a})}|0\rangle$ is a coherent state, where $|0\rangle$ is the lowest energy eigenstate of the oscillator.

5. A bead of mass m slides without friction on a straight wire of length a between two rigid walls (one wall at $x = 0$, the other at $x = a$).

a. Apply Schrödinger's equation to the one-dimensional problem to obtain the energy levels for the system.

b. If the normalized eigenfunctions for the system are $|n\rangle \equiv \psi_n = \sqrt{2/a} \sin \frac{n\pi x}{a}$, calculate the ratio of the probabilities that various energy states are occupied if a measurement indicates that the bead is precisely at the center of the wire. (Hint: The Dirac δ -function is useful.)

c. If a later measurement shows that the bead is not on the right half of the wire, but is anywhere on the left half with equal probability, what would be its normalized wavefunction?

d. Under the circumstance given in part c., what is the probability the system is in its lowest energy states $m = 1, 2$, and 3 ?

6. A particle of mass m moves in the one-dimensional potential $V(x)$:

$$V(x) = \begin{cases} \frac{1}{2}kx^2 & x \geq 0 \text{ (Note: } k > 0\text{)} \\ = \infty & x < 0 \end{cases}$$

a. What is the quantum mechanical equation of motion?

b. Find the energy eigenvalues and calculate the wavefunctions for the lowest two energy levels.

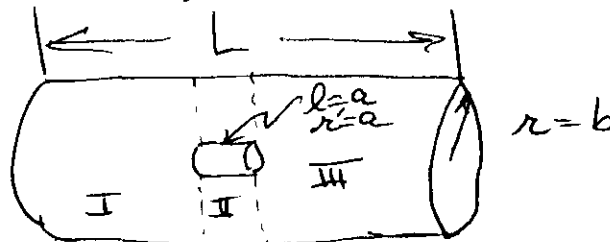
c. Suppose the system is perturbed by the time-dependent potential $U(x,t)$. If the system is initially in the first excited state, what is the amplitude for it to be in an arbitrary state for small times t ?

d. If the perturbation in part c. is independent of time, what is the decay exponential in t for small t ? Explain.

Part III: Electricity & Magnetism

Instructions: Work five (5) of the following six problems. Problems are to be worked on separate sheets of paper to facilitate grading. Each problem should take approximately 30 minutes to solve. Don't spend too much of your time on any one problem. Distribute your time wisely.

1. A solid cylindrical conductor of length L , radius b , has a conductivity σ_1 . At the center of the conductor is a defect consisting of another small cylinder of length a , radius a , and conductivity σ_2 .



- a. Find the resistance R of the conductor in terms of L , σ_1 , and b if $\sigma_1 = \sigma_2$.
- b. Estimate the change δR in resistance to first order in $\sigma_1 - \sigma_2$ ($\sigma_1 \neq \sigma_2$) by dividing the cylinder into three regions as shown above and treating the defect resistance R_d in region II to be in parallel with the remaining other resistance R_0 in region II. (Assume $b \gg a$.)
- c. Now replace the small defect cylinder by a small sphere of radius a . Let b and L approach ∞ . The necessary boundary conditions at $r = a$ are: (1) $\sigma_1 E_{1\perp} = \sigma_2 E_{2\perp}$ and (2) $E_{1\theta} = E_{2\theta}$. Use the electric potentials $\Phi_1(r, \theta) = \frac{-j_0}{\sigma_1} (r + \frac{c}{r^2}) \cos \theta$ and $\Phi_2(r, \theta) = Dr \cos \theta$, along with $E_\theta = -\frac{1}{r} \frac{\partial \Phi}{\partial \theta}$ to show that $D = \frac{-3j_0}{\sigma_2 + 2\sigma_1}$. [End of problem]
- (From this, one can find Φ_2 and j_2 , the current density inside the defect region.)

2. a. An insulating circular ring (radius R) lies in the xy plane, centered at the origin. It carries a linear charge density $\lambda = \lambda_0 \sin \phi$, where λ_0 is constant and ϕ is the usual azimuthal angle. The ring is now set spinning at constant angular velocity ω about the z axis. Calculate the power radiated.

- b. Now suppose you take a plastic ring of radius R and glue charge on it, so that the line charge density is $\lambda_0 |\sin(\theta/2)|$. Find the exact scalar and vector potentials at the center of the ring if you spin it about its axis at an angular velocity ω .

Part III: Electricity & Magnetism p. 2

3. A spherical shell of radius R , carrying a uniform surface charge σ , is set spinning at angular velocity ω . Show that the vector potential it produces is:

$$\mathbf{A} = \begin{cases} \frac{\mu_0 R \omega \sigma}{3} r \sin \theta \hat{\phi}, & r \leq R \\ \frac{\mu_0 R^4 \omega \sigma}{3} \frac{\sin \theta}{r^2} \hat{\phi}, & r \geq R. \end{cases}$$

What is the magnetic field inside and outside the spherical shell?

4. The free space retarded solution to the three-dimensional scalar wave equation

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G^{3D}(\vec{x}, t; \vec{x}', t') = -4\pi c^2 \delta^{(3)}(\vec{x} - \vec{x}') \delta(t - t'),$$

is given by

$$G^{3D}(\vec{x}, t; \vec{x}', t') = \frac{\delta(\tau - \frac{R}{c})}{R},$$

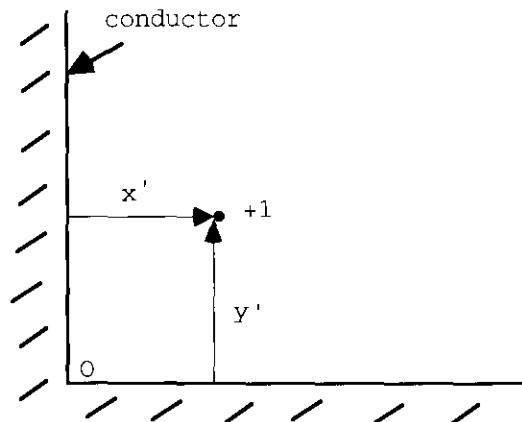
where $R = |\vec{x} - \vec{x}'|$, $\tau \equiv t - t'$ and $\delta(x)$ is the Dirac delta function.

a. Show that the two-dimensional Green function is given by

$$G^{2D}(\vec{x}_\perp, t; \vec{x}'_\perp, t') = \int_{-\infty}^{\infty} dz G^{3D}(\vec{x}, t; \vec{x}', t').$$

b. Given the above explicit form for $G^{3D}(\vec{x}, t; \vec{x}', t')$, carry out the integration in part a. to find $G^{2D}(\vec{x}_\perp, t; \vec{x}'_\perp, t')$.

5. a. Find the Dirichlet electrostatic Green function, $G(\vec{x}; \vec{x}')$, for the (three-dimensional) geometry shown. The walls are perfect conductors.



The angle between the infinitely long conducting planes is exactly 90° . (Take your coordinate origin at 0.)

b. Find the work done, W , in removing the $+1$ charge to spatial infinity.

Part III: Electricity & Magnetism p. 3

6. A parallel-plate capacitor has a plate area which is much much larger than the separation between the plates, d . Between these plates is a dielectric (of $\epsilon > \epsilon_0$) except for small spherical air bubble (of radius $R \ll d$) which is centered between the plates. The plates are connected to a potential difference of V_0 volts. Assuming that we never look close to a plate edge:

- a. Determine a reasonable approximation for the electric field between the plates.
- b. Make a rough sketch of the electric field lines.

Part IV: General Physics, Mathematical Physics, & Statistical Mechanics

Instructions: Work five (5) of the following six problems. Problems are to be worked on separate sheets of paper to facilitate grading. Each problem should take approximately 30 minutes to solve, with the exception of number one. Don't spend too much of your time on any one problem. Distribute your time wisely.

1. (25 minutes) A system consists of N distinguishable atoms, each of which has available energy levels: $E_n = nE_0$ for all $n = 0, 1, 2, \dots, \infty$. Each energy level has a degeneracy of: $g_n = g_0/n!$. Show that the partition function for the system is

$$Z = g_0^N \exp\{N e^{-(E_0/kT)}\}.$$

Calculate the internal energy (U) and heat capacity (C_V) of the system.

2. Consider a particle of mass m that is free to move in a one-dimensional region of length L that closes on itself, for instance, a bead which slides frictionlessly on a circular wire of circumference L .

- a. Show that the stationary states can be written in the form

$$\psi_n(x) = \frac{1}{\sqrt{L}} \exp\left[\frac{2\pi i n x}{L}\right] \quad (-L/2 < x < L/2)$$

where $n = 0, \pm 1, \pm 2, \dots$ and the allowed energies are

$$E_n = \frac{2}{m} \left(\frac{n\pi\hbar}{L}\right)^2$$

What are the degeneracies of these states?

- b. Introduce the perturbation

$$H' = -V_0 \exp[-x^2/a^2]$$

where $a \ll L$. (This puts a "dimple" in the potential at $x = 0$, as though we bent the wire slightly to make a "trap.") Find the first-order correction to E_n .

3. The energy of a system is a function of some N generalized coordinates, q_i , and their corresponding N generalized momenta, p_i , i.e. $E = E(q_1, q_2, \dots, q_N, p_1, p_2, \dots, p_N)$. The energy can also be written in the form $E = e(p_i) + \mathcal{E}(q_1, \dots, q_N)$ where $e(p_i) = bp_i^3$ depends only on p_i and the constant b , whereas $\mathcal{E}(q_1, \dots, q_N)$ is known to be independent of p_i . DERIVE a simple expression for the mean value of $e(p_i)$, in terms of kT under the assumption that the system is in thermal equilibrium (k is Boltzmann's constant and T is the temperature).

4. A two-dimensional nonrelativistic gas is confined to a finite area, S . The energy and volume per area of the gas are given by

$$\frac{E}{S} = \sum_{\vec{p}} \epsilon_{\vec{p}} \langle n_{\vec{p}} \rangle, \text{ and } \frac{N}{S} = \sum_{\vec{p}} \langle n_{\vec{p}} \rangle,$$

$$\text{where } \epsilon_{\vec{p}} = \frac{\beta p^2}{2m} \left(\beta = \frac{1}{kT} \right) \text{ and } \langle n_{\vec{p}} \rangle = \begin{cases} \frac{1}{\exp(\alpha + \frac{\beta p^2}{2m}) + 1}, & \text{fermions} \\ \frac{1}{\exp(\alpha + \frac{\beta p^2}{2m}) - 1}, & \text{bosons} \end{cases}.$$

a. Consider the fermion case at $T = 0$ ($\alpha = -\mu\beta$). Show that

$$\frac{E}{S} = C \left(\frac{N}{S} \right)^2.$$

Find the constant of proportionality, C .

b. Consider the boson case at small but finite temperature T ($T \ll \frac{h^2}{2mk} \left(\frac{N}{S} \right)$).

Show that $\frac{E}{S}$ is, in contrast to fermions, independent of $\frac{N}{S}$ and varies with temperature as $\frac{E}{S} \sim T^2$. (Note: There is no Bose-Einstein condensation at low temperatures in two dimensions.)

5. Use complex variables to evaluate the integral

$$\int_0^{\infty} \left[\frac{\ln^2 x}{1+x^2} \right] dz.$$

Be careful to show your contour of integration and to justify your procedure.

6. The Green function for the two-dimensional Laplace PDE may be obtained in series form.

a. Obtain the solution of the two-dimensional PDE $\nabla^2 G = \delta(\mathbf{r} - \mathbf{r}_0)$ in the form of a complex Fourier series, $G(\mathbf{r} | \mathbf{r}_0; \theta | \theta_0) = \sum_{n=-\infty}^{n=+\infty} g_n(\mathbf{r}, \mathbf{r}_0; \theta_0) e^{in\theta}$, by solving the differential equation for functions g_n . [Hints: The case $n = 0$ should be treated separately. Also, $\delta(\mathbf{r} - \mathbf{r}_0) = \frac{1}{r} \delta(r - r_0) \delta(\theta - \theta_0)$]

b. Having obtained $G(\mathbf{r} | \mathbf{r}_0; \theta | \theta_0)$, set it equal to $(1/2\pi) \log |\mathbf{r} - \mathbf{r}_0|$ and establish the result:

$$\log \sqrt{r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)} = \begin{cases} \log r_0 - \sum_{n=1}^{\infty} \left(\frac{r}{r_0} \right)^n \frac{1}{n} \cos n(\theta - \theta_0), & r < r_0, \\ \log r - \sum_{n=1}^{\infty} \left(\frac{r_0}{r} \right)^n \frac{1}{n} \cos n(\theta - \theta_0), & r > r_0. \end{cases}$$

BAYLOR

U N I V E R S I T Y

1997 Ph.D. Preliminary Examination

Part I: Mechanics

Time: 3 hours

Date: May 29, 1997

Instructions: Work five (5) of the following seven problems. Work each problem on a separate sheet of paper and turn in only the five problems you wish to have graded. Each problem is designed to require approximately 30 minutes to solve. Use your time wisely and be careful to not spend too much time on any one problem. **Good luck!**

1. A spherical pendulum is a pendulum that is unconstrained in its angular motion; it can move freely about its pivot point. Consider a spherical pendulum of length L acted on by gravity with a mass M attached to its end. The mass of the attachment of length L is negligible.
 - a. Derive the Euler-Lagrange equations for the pendulum in the spherical coordinates θ (polar angle) and ϕ (azimuthal angle). Take $\theta=0$ to represent the equilibrium position. Show that the z -component of the angular momentum, l_z , is conserved.
 - b. For uniform circular motion in ϕ when $\theta = \text{const.} \neq 0$, show that the period of the motion may be written as

$$T = 2\pi \left(\frac{L \cos \theta}{g} \right)^{1/2}.$$

- c. Consider general motions of the pendulum subject only to the condition that $\theta \ll \pi/2$. Defining the total energy of the system, E , relative to $\theta=0$, show that the turning points of the θ motion are

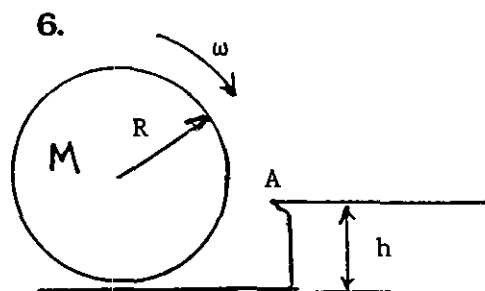
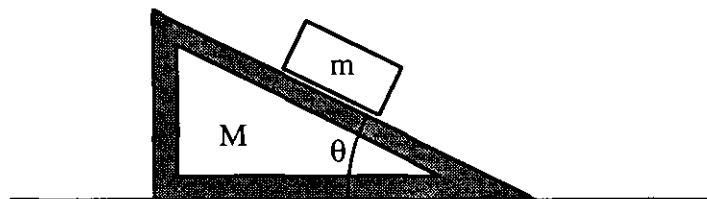
$$\theta_t = \sqrt{\left(\frac{E}{MgL} \right) \pm \sqrt{\left(\frac{E}{MgL} \right)^2 - \frac{l_z^2}{gM^2L^3}}}.$$

2. A thin hoop of radius R and mass M oscillates in its own plane with one point of the hoop fixed. Attached to the hoop is a small mass M constrained to move (in a frictionless manner) along the hoop. Considering only small oscillations, show that the eigenfrequencies are

$$\omega_1 = (2g/R)^{1/2}, \quad \omega_2 = (g/2R)^{1/2}.$$

- b. Find the two sets of initial conditions that allow the system to oscillate in its normal modes. Describe the physical situation for each mode.

3. A cannon in a fort overlooking the ocean fires a shell of mass M at an angle of 45° and muzzle velocity v_0 . At the highest point the shell explodes into two fragments (mass $m_1 + m_2 = M$), with an additional energy E , traveling in the original horizontal direction. Find the distance separating the two fragments when they land in the ocean.
4. Two point masses m are connected by a weightless string of length L which passes through a small hole in a horizontal table. One mass moves on the surface of the table and the other mass moves vertically up and down. There is no friction between the table and the mass on the table. Write Lagrange's equations of motion for the mass on the table. (Use the length of the string on the table as one generalized coordinate and the angle of the mass on the table relative to the horizontal as the other generalized coordinate.)
5. A particle of mass m is sliding on a smooth wedge of mass M and wedge angle θ , which itself is free to slide on a smooth horizontal surface, as shown in the figure. Obtain the Lagrangian, equations of motion, and find the acceleration of the particle.

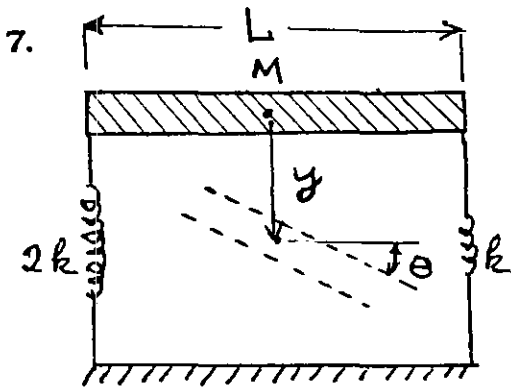


6. A homogeneous circular disk of mass M , radius R , and angular velocity ω rolls across a horizontal floor and collides with a fixed, raised step of height h ($h < R$). The collision is strictly inelastic and of short duration. Assume that the disk doesn't slip on the edge.

- (a) Show that the minimum angular velocity ω_0 for the disk to "roll up on the step" is

$$\omega_0 = \frac{2\sqrt{gh/3}}{R(1 - 2h/3R)}$$

- (b) What would ω_0 be if the rolling object were a homogeneous sphere of mass M and radius R instead of a uniform disk? (Use $I_{\text{sph}} = (2/5)MR^2$.)



A uniform bar of mass M and length L is supported in a horizontal position on top of two vertical springs as shown in the figure to the left. The spring on the left has a spring constant $2k$, while that on the right has a spring constant k . Assume that the motion is confined to a vertical plane as shown in the diagram.

Use the variables y and θ as shown in the figure to set up the kinetic energy and potential energy matrices \mathbb{T} and \mathbb{V} for the system, and then solve to find the two eigenfrequencies for the system. Express your answer in terms of k , M , and numerical values.

BAYLOR

UNIVERSITY

1997 Ph.D. Preliminary Examination Part II: Quantum Mechanics

Time: 3 hours
Date: May 29, 1997

Instructions: Work five (5) of the following seven problems. Work each problem on a separate sheet of paper and turn in only the five problems you wish to have graded. Each problem is designed to require approximately 30 minutes to solve. Use your time wisely and be careful to not spend too much time on any one problem. **Good luck!**

1. Solve the time-independent one-dimensional Schrödinger equation for a delta-function potential well centered at the origin [$V(x) = -\alpha \delta(x)$, with $\alpha > 0$]. Find the energy and (normalized) eigenfunction of the (single) bound state.
- b. Using the Gaussian trial function, $\psi(x) = A e^{-bx^2}$, find the best estimate for the bound state of the delta-function potential. How does the estimate compare with the exact result in part (a.)?

2. Given a unitary transformation, U , that is not time dependent, and given an operator A such that $[A, B] \neq AB - BA$

$$\frac{dA}{dt} = \frac{1}{i\hbar} [A, H],$$

and defining

$$A' \neq UAU^{-1},$$

show that

$$\frac{dA'}{dt} = \frac{1}{i\hbar} [A', H'],$$

where

$$H' = UHU^{-1}.$$

- b. Again given

$$\frac{dA}{dt} = \frac{1}{i\hbar} [A, H], \quad A' \neq UAU^{-1},$$

but now assuming that U is time dependent, show that

$$\frac{dA'}{dt} = \frac{1}{i\hbar} [A', K],$$

where

$$K = UHU^{-1} - i\hbar \frac{dU}{dt} U^{-1}.$$

3. Given the eigenvalue-eigenvector statement for the Hamiltonian, $H(\lambda)$, and energy, $E(\lambda)$,

$$H(\lambda) |\psi(\lambda)\rangle = E(\lambda) |\psi(\lambda)\rangle,$$

where λ is some parameter in the problem, show that

$$\langle \psi(\lambda) | \frac{\partial H}{\partial \lambda} | \psi(\lambda) \rangle = \frac{\partial E}{\partial \lambda}.$$

(This is called the Feynman-Hellmann theorem.)

- b. Now, given that the hydrogen atom energy values are given by

$$E_n = -\frac{mZe^4}{2h^2n^2},$$

and the Hamiltonian is

$$H = \frac{\vec{p}^2}{2m} - \frac{Ze^2}{r},$$

show in a hydrogen atom energy eigenstate, $|n\rangle$, that

$$\langle n | \frac{1}{r} | n \rangle = \frac{mZe^2}{h^2n^2}.$$

4. Assume that in the lab you have devised a device allowing the measurement of the spin component of an electron in an arbitrary direction. When you use your device to make such a measurement on a certain electron with respect to a certain direction (e_2), the result is spin up (\uparrow) in the direction of e_2 .
- a. In which state is the electron after this measurement? Draw the state vector in two-dimensional (complex) spin space.
- b. Let the spin component of the same electron now be measured in another direction ($e_{2'}$) which is inclined to the original direction by an angle of $\theta = 90^\circ$, $\theta = 180^\circ$. What is the probability of the result being spin \uparrow or spin \downarrow ? What are the positions of the vectors spin \uparrow and spin \downarrow in the z' direction in spin space?
- c. By what angle are they rotated compared with spin \uparrow and spin \downarrow in the z direction? What does the above tell you about the relationship between configuration space and spin space? (This is important to remember, especially in the case of spin 1 particles where both spaces have three dimensions!)

5. Consider a particle in an infinite square well described initially by a wave that is a superposition of the ground and first excited states of the well:

$$\Psi(x, 0) = C [\psi_1(x) + \psi_2(x)].$$

- a. Show that the value $C = 1/\sqrt{2}$ normalizes this wave, assuming ψ_1 and ψ_2 are themselves normalized.
- b. Find $\Psi(x, t)$ at any later time t .
- c. Show that the superposition is not a stationary state, but the average energy in this state is the arithmetic mean $(E_1 + E_2)/2$ of the ground and first excited state energies E_1 and E_2 .
- d. For this nonstationary state, show that the average particle position $\langle x \rangle$ oscillates with time as

$$\langle x \rangle = x_0 + A \cos(\Omega t)$$

where

$$x_0 = \left(\int x |\psi_1|^2 dx + \int x |\psi_2|^2 dx \right)$$

$$A = \int x \psi_1^* \psi_2 dx$$

and

$$\Omega = (E_2 - E_1)/\hbar.$$

6.

Given the Hermitian matrix $\Gamma = \begin{pmatrix} \gamma & 0 & i\beta\gamma \\ 0 & 1 & 0 \\ -i\beta\gamma & 0 & \gamma \end{pmatrix}$

- (a) Find the eigenvalues of Γ .
- (b) Find the unitary matrix U that will diagonalize Γ .
- (c) Verify that $UU^\dagger = \mathbb{1}$
- (d) Verify that $U^{-1}\Gamma U$ is diagonal.

(Partial ans.: (b) $U = \begin{pmatrix} 1/\sqrt{2} & 0 & - \\ 0 & 1 & - \\ - & - & - \end{pmatrix}$)

7. (a) Prove that $(E_\ell - E_k) \langle k|x|\ell \rangle = \frac{i\hbar}{m} \langle k|p_x|\ell \rangle$ for $H = \frac{p^2}{2m} + V(\vec{r})$ by use of $E_\ell|\ell\rangle = H|\ell\rangle$ and $E_k\langle k| = \langle k|E_k = \langle k|H$ (Dirac notation using "bras" and "kets.")
- (b) Use this and the closure property to prove the following sum rule:

$$\sum_k (E_k - E_\ell) |\langle k|x|\ell \rangle|^2 = \frac{\hbar^2}{2m}. \quad (\text{Hint: } E_k - E_\ell = \frac{1}{2}(E_k - E_\ell) - \frac{1}{2}(E_\ell - E_k)).$$

BAYLOR

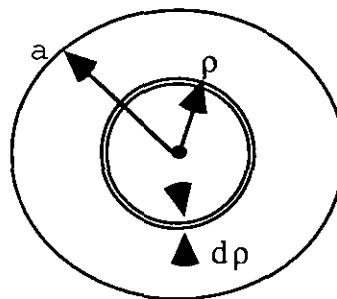
U N I V E R S I T Y

1997 Ph.D. Preliminary Examination Part III: Electricity & Magnetism

Time: 3 hours
Date: May 30, 1997

Instructions: Work five (5) of the following seven problems. Work each problem on a separate sheet of paper and turn in only the five problems you wish to have graded. Each problem is designed to require approximately 30 minutes to solve. Use your time wisely and be careful to not spend too much time on any one problem. **Good luck!**

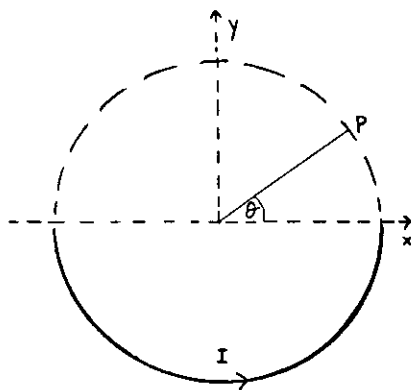
1. A point electric charge of value q is embedded in the planar boundary of two non-conducting dielectric materials of dielectric constants ϵ_1 and ϵ_2 . Find the \mathbf{E} and \mathbf{D} fields on both sides of the boundary. [Hint: Be careful! One way to proceed is to find the answer for $d \neq 0$, where d is the distance of the charge from the boundary, and then take the limit $d \rightarrow 0$.]
2. A long straight wire with a circular cross-section carries a uniform current density along the z -axis, as shown. The total current flowing is $I = \text{const.}$ in time.



(z -axis is up)

- a. Find the differential radial pressure, $dP(\rho) \neq \frac{d\vec{F} \cdot \hat{\rho}}{d\rho}$, exerted on a tube of material (thickness $d\rho$) by the current at a distance ρ from the center of the wire.
- b. Integrate your expression to find the total pressure, $P(\rho)$, exerted at the distance ρ from the center of the wire. Is this pressure inward or outward?

3. An eccentric hole of radius a is bored parallel to the axis of a right circular cylinder of radius b ($b > a$). The two axes are at a distance d apart. A current of I amperes flows in the cylinder. What is the magnetic field at the center of the hole?
- b. A semicircular wire carries a steady current I (it must be hooked up to some *other* wires to complete the circuit, but disregard them here). Find the magnetic field at a point on the other semicircle (see figure).



4. A point charge q is located a distance L from the center of a grounded, conducting sphere of radius a ($a < L$).
- a. Find the potential at any point exterior to the sphere (where you can exclude the point at which the charge is located).
- b. Calculate the charge distribution on the surface of the sphere. **HINT:** Use a Legendre polynomial expansion with the origin at the center of the sphere and the polar axis passing through the point charge.
- c. Now solve this problem using the method of images and show that your results agree with what you obtained above.
5. A parallel-plate capacitor consists of two circular plates, so that the system has an axis of symmetry. The radius is a , the plate separation is l , and the material filling the space between the plates has dielectric constant ϵ . The capacitor is charged by being placed in a circuit that contains a source of emf V_0 and a series resistor R . If the circuit is completed at time $t = 0$, find the following quantities within the capacitor as functions of time (Neglect edge effects and assume $RC \gg a/c$.)
- a. the electric field,
 b. the magnetic field,
 c. Poynting's vector,
 d. the field energy,
 e. the scalar potential, and
 f. the vector potential.

6. A plane-polarized electromagnetic wave is incident on a sphere composed of a solid material and having a radius a . You can assume that the wavelength λ is large compared with the radius of the sphere.
- Find the electric field at any instant of time throughout the sphere in terms of the external (applied) field E and the polarizability of the material, α .
 - Assuming that the dipole moment per unit volume of the sphere is simply proportional to the internal electric field, find the total cross section for scattering the radiation.
7. Show that the equation of the lines of electrical force between two parallel line charges, of strength q and $-q$ per unit length, at $x = a$ and $x = -a$, in terms of the electric flux per unit length N , between the line of force and the positive x -axis, is (use SI units $q/4\pi\epsilon r^2$ for a point charge--you determine what it is for a line charge):

$$\left[y - a \cot\left(\frac{2\pi N}{q}\right) \right]^2 + x^2 = a^2 \csc^2\left(\frac{2\pi N}{q}\right) .$$

Hint: $\tan(a \pm b) = (\tan a \pm \tan b) / [1 \mp \tan a \tan b]$

BAYLOR

UNIVERSITY

1997 Ph.D. Preliminary Examination

Part IV: General, Mathematical and Statistical Physics

Time: 3 hours

Date: May 30, 1997

Instructions: Work five (5) of the following seven problems. Work each problem on a separate sheet of paper and turn in only the five problems you wish to have graded. Each problem is designed to require approximately 30 minutes to solve. Use your time wisely and be careful to not spend too much time on any one problem. **Good luck!**

1. $\mathbf{D} |u_i\rangle = \sum_j D_{ji}^u |u_j\rangle$ where $\{|u_i\rangle\}$ is an orthonormal basis and $D^u = \begin{pmatrix} 4 & i\sqrt{3} \\ -i\sqrt{3} & 2 \end{pmatrix}$.
 - a. Find all solutions of $\mathbf{D} |v\rangle = \lambda |v\rangle$ (Hint: the eigenvalues are integers.)
 - b. Introduce a new orthonormal basis $\{|v_i\rangle\}$ and show explicitly how to get from $\{|u_i\rangle\}$ to $\{|v_i\rangle\}$ for \mathbf{D} . Do the matrix multiplication necessary to get D^v .
 - c. Next, consider the perturbation $W = \epsilon \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ to D^u . Find the first order corrections to the eigenvalues and eigenvectors.
 - d. Compare the corrected and actual eigenvalues for the case $\epsilon = 0.1$.
2. Given the expressions for the pressure, P , and internal energy, U , of a nonrelativistic ideal Fermi-Dirac gas,

$$P = g \frac{kT}{h^3} \int d^3p \ln(1 + e^{\beta(\mu - \epsilon_p)}),$$

$$U = g \frac{V}{h^3} \int d^3p \frac{\epsilon_p}{1 + e^{\beta(\epsilon_p - \mu)}},$$
$$\epsilon_p = \frac{p^2}{2m},$$

where the integrations are over all momentum values, show, without actually doing the integrals (which are very difficult), that

$$U = \frac{3}{2} PV.$$

[Hint: Integrate by parts.]

3. Stephen Hawking calculated the entropy of a non-rotating, uncharged Black Hole and obtained the expression

$$S = \pi k_B c^3 A / 2Gh$$

where A (the area of the Black Hole) = $4\pi R^2$ with R (the Schwarzschild radius) = $2GM/c^2$. Here M is the mass of the star and G is Newton's gravitational constant. Show that this equation implies that the temperature T of a Black Hole is given by

$$1/T = 16\pi^2 Gk_B M / hc^3.$$

- b. If Black Holes radiate according to the Stefan-Boltzmann law they lose mass. Show that the rate change of mass (dM/dt) is given by the equation

$$c^2 (dM/dt) = -A\sigma T^4.$$

Solve this equation to obtain the variation in the mass of a Black Hole as a function of time. Give that the initial mass, $M(0)$, of the Black Hole is about 2×10^{11} kg, show that the Black Hole evaporates in a time of $t = 7 \times 10^{17}$ s, which is roughly the age of the Universe.

4. An assembly of N particles of spin $1/2$ are lined up on a straight line. Only nearest neighbors interact. When the spin of neighbors are both up or both down, their interaction energy is ϵ . When one is up and one is down, the interaction energy is $-\epsilon$. Show that the partition function, Z , of the assembly at temperature T is given by

$$Z = 2^N [\cosh(\epsilon/kT)]^{N-1}.$$

What is the entropy of the system?

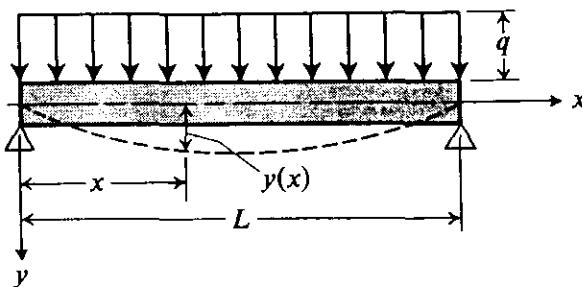
5. Show that the generators L_i of a unitary matrix group, i.e. $U^{-1} = U^\dagger$ for each group element U , can be chosen in such a way that the relation $\text{Tr}(L_i L_j) = \delta_{ij}/2$ is valid.
- b. Prove that the resulting structure constants are purely imaginary and totally antisymmetric in this case.

6. A simply supported beam is shown in the figure below. (Note the direction of the y-axis, which was chosen to make the deflection $y(x)$ positive.) The beam is loaded by a variable load per unit length $q(x) = ax/L$. The deflection $y(x)$ is known to satisfy the differential equation

$$\frac{d^4 y}{dx^4} = \frac{1}{EI} q(x)$$

where $1/EI$ is the rigidity of the beam.

- a. Using a suitable Fourier series, solve for $y(x)$.
- b. Solve the problem in closed form by integrating the DE. (Recall the free end BC $y''(0) = y''(L) = 0$.) Show that the error for the deflection at the midpoint is less than 0.4% when only the first Fourier term is used.



7. Show that if stimulated emission is neglected, leaving only two Einstein coefficients, an appropriate relation between these coefficients will be consistent with the thermal equilibrium between the atom and a radiation field expected of a Wien spectrum, but *not* of a Planck spectrum.
- b. Derive the relation between the Einstein coefficients by imagining the atom to be in thermal equilibrium with a neutrino field (spin 1/2) rather than a photon field (spin 1). **HINT:** Remember that neutrinos are Fermi-Dirac particles and thus obey the exclusion principle. In addition, their equilibrium intensity is given by

$$I_\nu = \frac{2h\nu^3/c^2}{\exp(h\nu/kT) + 1}$$

BAYLOR

U N I V E R S I T Y

1998 Ph.D. Preliminary Examination Part I: Mechanics

Time: 3 hours
Date: May 28, 1998

Instructions: Work five (5) of the following seven problems. Work each problem on a separate sheet of paper and turn in only the five problems you wish to have graded. Each problem is designed to require approximately 30 minutes to solve. Use your time wisely and be careful to not spend too much time on any one problem. **Good Luck!**

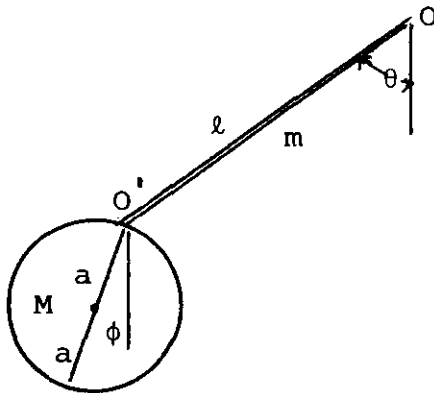
1. Consider a linear chain in which alternate ions have mass M_1 and M_2 , and only nearest neighbors interact. If the interaction between nearest neighbors is characterized by a spring constant K and the distance is a , find the normal modes of this one dimensional crystal.
2. A uniform rectangular block of dimensions $2a$ by $2b$ by $2c$ and mass m spins about a long diagonal with angular rate ω . Find
 - (a) the inertia tensor for a coordinate system with origin at the center of the block and with axes normal to the face.
 - (b) the angular momentum about the origin.
 - (c) the rotational kinetic energy of the block.
3. A mass M moves horizontally along a smooth rail. A pendulum is hung from M with a weightless rod and mass m at its end. Find the eigenfrequencies of small oscillations of this system. Determine the normal coordinates and describe the normal modes.
4. A train at latitude λ in the northern hemisphere is moving due north with a speed v along a straight and level track. Which rail experiences the larger vertical force? Show that the ratio R of the vertical forces on the rails is given by

$$R = 1 + \frac{8\Omega v h \sin \lambda}{ga}$$

where h is the height of the center of gravity of the train above the rails which are at a distance a apart, g is the acceleration due to gravity, and Ω is the angular velocity of the Earth's rotation.

5. Determine the frequency of small vibrations of a uniform hemisphere which lies on a smooth horizontal surface in a uniform gravitational field.

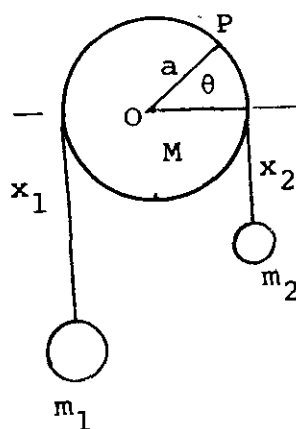
6.



A plane pendulum consists of a uniform rod of length l and mass m , suspended in a vertical plane at one end at O . At the other end of the rod, at O' , is a uniform disk of mass M and radius a , attached so that it can pivot (rotate) freely in the same vertical plane. Use the angles θ and ϕ as shown in the accompanying figure to the left to set up the Lagrangian and from the Lagrangian determine the two second-order differential equations of motion.

(Use $I_{\text{disk}} = \frac{1}{2}Ma^2$; $I_{\text{rod}}(\text{about center}) = (1/12)m l^2$)

7.



A pulley of mass M , radius a , is used in an Atwood machine to support two masses m_1 and m_2 as shown. Assume there is no friction. Use the method of Lagrange undetermined multipliers to find the acceleration of the two masses and the tension T in the string.

BAYLOR

U N I V E R S I T Y

1998 Ph.D. Preliminary Examination
Part II: Quantum Mechanics

Time: 3 hours
Date: May 28, 1998

Instructions: Work five (5) of the following seven problems. Work each problem on a separate sheet of paper and turn in only the five problems you wish to have graded. Each problem is designed to require approximately 30 minutes to solve. Use your time wisely and be careful to not spend too much time on any one problem. **Good Luck!**

1. The Hamiltonian of a particle can be expressed in the form

$$H = E_1 a^\dagger a + E_2 (a + a^\dagger)$$

where E_1 and E_2 are constants, and the operators a and its adjoint a^\dagger satisfy the commutation relation $[a, a^\dagger] = 1$. Find the energy eigenvalues. (HINT: Use a suitable transformation to convert H into a form similar to that of the harmonic oscillator.)

2. (a) Use the variational principle to show that an attractive square-well potential in one dimension, no matter how weak, produces at least one bound state. Take the trial wavefunction to be $\exp(-a^2 x^2)$.

(b) In a given problem, the energy of a stationary state is estimated to be E_{WKB} by the WKB approximation, and E_{var} by the variational method. If $E_{\text{WKB}} > E_{\text{var}}$, which result will you consider to be more reliable and why? What if $E_{\text{var}} > E_{\text{WKB}}$?

3. The Schrödinger equation for an $l = 0$ state with the exponential potential

$$V(r) = -\frac{a^2}{8} e^{-r/r_0}$$

can be solved analytically and has the solution $\psi = \frac{u(r)}{r}$ with

$$u(r) = J_n(ar_0 e^{-r/2r_0})$$

where J_n is the Bessel function of order n , and

$$E = -\frac{1}{8r_0^2} n^2$$

a. Prove by direct substitution that the given function solves the Schrödinger equation.

b. $J_{1/2}(x)$ has its first zero at π . Using the trial function,

$$\phi(r) = \frac{u(r)}{r} = e^{-\lambda/r},$$

calculate the variational estimate of the ground state energy for $n = 1/2$.

4. Given the (one-dimensional) harmonic oscillator eigenvalue equation,

$$H|n\rangle = E_n|n\rangle,$$

and Hamiltonian,

$$H = \frac{p_x^2}{2m} + \frac{1}{2} m\omega^2 x^2,$$

(a) Derive the position space energy eigenvalue equation for $u_n(x') \equiv \langle x' | n \rangle$ ("the Schrödinger equation in position space").

(b) Derive the momentum space energy eigenvalue equation for $v_n(p'_x) \equiv \langle p'_x | n \rangle$ ("the Schrödinger equation in momentum space").

(c) Find a simple substitution of variables which changes the differential equation in (a) into the equation in (b); express $v_n(p'_x)$ in terms of $u_n(x')$ outside of an overall normalization.

5. A bead of mass m slides without friction on a straight wire of length a between two rigid walls (one wall is at $x = 0$, the other at $x = a$).

- (i) What are the energy levels of this system?
- (ii) Show explicitly that the wave functions corresponding to different energies are orthogonal.
- (iii) If a measurement indicates that the bead is exactly at the middle of the wire, show that the probability of finding the particle in the n^{th} state is $2/a$ for odd n and zero for even n .

6. The following problem pertains to the algebra of the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where the last matrix is the unit matrix whose trace is 2.

(a) Prove the following algebraic relationship:

$$(\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B}) = \mathbb{1} \vec{A} \cdot \vec{B} + i \vec{\sigma} \cdot (\vec{A} \times \vec{B})$$

(b) Then prove that the commutator of $\vec{\sigma} \cdot \vec{A}$ and $\vec{\sigma} \cdot \vec{B}$ is:

$$[\vec{\sigma} \cdot \vec{A}, \vec{\sigma} \cdot \vec{B}] = 2i \vec{\sigma} \cdot (\vec{A} \times \vec{B})$$

(c) Use the result in part (a) to evaluate $(\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B})(\vec{\sigma} \cdot \vec{C})$ in the form of $a + \vec{b} \cdot \vec{\sigma}$, i.e., find the scalar a and vector \vec{b} in terms of scalar and vector products of \vec{A} , \vec{B} , and \vec{C} .

7. The quantum-mechanical representation of the angular momentum \vec{L} is given by $\vec{L} = \vec{r} \times \vec{p}$, except one must be careful about the algebra of \vec{r} and \vec{p} . Use various commutator relations such as

$$[\vec{p}_i, \vec{r}_j] = (\hbar/i)\delta_{ij}$$

to prove the following alternate quantum-mechanical form:

$$\vec{L}^2 = \sum_i \sum_j (r_i p_j r_i p_j - r_i p_j r_j p_i) = \vec{r}^2 \vec{p}^2 - (\vec{r} \cdot \vec{p})^2 + i\hbar(\vec{r} \cdot \vec{p}).$$

Hint: Use the vector identity $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{C} \cdot \vec{A}) - \vec{C}(\vec{A} \cdot \vec{B})$

$$\text{and } \vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

BAYLOR

U N I V E R S I T Y

1998 Ph.D. Preliminary Examination
Part III: E&M

Time: 3 hours
Date: May 29, 1998

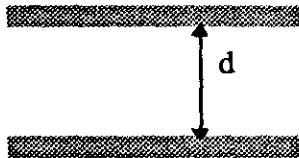
Instructions: Work five (5) of the following seven problems. Work each problem on a separate sheet of paper and turn in only the five problems you wish to have graded. Each problem is designed to require approximately 30 minutes to solve. Use your time wisely and be careful to not spend too much time on any one problem. **Good Luck!**

1. Consider two infinite concentric conducting cylinders of radii R_1 and R_2 ($R_1 < R_2$). Solve the differential equation $\nabla^2\phi(\vec{r})=0$ under the Dirichlet's boundary value condition in the domain $R_1 < r < R_2$. Compute capacities and coefficients of inductors, defined by

$$Q_j = \sum_{k=1}^2 C_{jk} V_k \quad (j = 1, 2)$$

of the system. Here Q_j is the total electric surface charge of the j -th cylinder (per unit length along the z -axis).

2. Consider a parallel plate wave guide with separation distance d as shown below. Compute explicit expressions for \vec{E} and \vec{B} of TEM and TE modes.



3.(a) A small magnetic dipole, $d\vec{m}$, located at \vec{x}' , gives rise to a magnetic scalar potential given by

$$d\Phi_M = \frac{d\vec{m} \cdot (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}.$$

Show that, for a volume magnetization, this leads to the expression (\hat{n}' is a surface normal)

$$\Phi_M(\vec{x}) = -\int d^3x' \frac{\vec{\nabla}' \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} + \int da' \frac{\hat{n}' \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|},$$

for the vector potential where the volume and surface contributions are explicitly separated.

(b) A uniform volume magnetization,

$$\vec{M} = M_0 \hat{k},$$

($M_0 = \text{const.}$) is established in a sphere of radius a . Find the magnetic field outside the sphere and show it is the potential of a dipole with dipole moment

$$\vec{m} = \frac{4\pi a^3}{3} \vec{M}.$$

[Hint: The Coulomb expansion is given by

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{l,m} \frac{r_{<}^l}{r_{>}^{l+1}} \frac{4\pi}{2l+1} Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi').$$

The first several $Y_{lm}(\theta, \phi)$ are given by:

$$\begin{aligned} Y_{00}(\theta, \phi) &= \frac{1}{\sqrt{4\pi}} \\ Y_{11}(\theta, \phi) &= -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi}, \\ Y_{10}(\theta, \phi) &= \sqrt{\frac{3}{4\pi}} \cos\theta, \\ Y_{1-1}(\theta, \phi) &= \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi}. \end{aligned}$$

4. Calculate the electric and magnetic shielding coefficients of a spherical shell of inner radius a and outer radius b . That is, first place the shell in a uniform electric field and calculate the ratio of the field at the center to the field at infinity. Then, do the same for a uniform magnetic field. Assume that the sphere is made of a material having dielectric constant ϵ and permeability μ .

Hints: In spherical coordinates, $\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial}{\partial r}) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$

Also, $P_l(x) = \frac{1}{2^l l!} \frac{d^l (x^2 - 1)^l}{dx^l}$ is a solution of the Legendre equation

$$\frac{d}{dx} [(1 - x^2) \frac{d}{dx} P_l(x)] = -l(l + 1) P_l(x)$$

5. A phonograph record of radius R , carrying a uniform surface charge σ , is rotating at constant angular velocity ω . Find

(a) the magnetic field at a point $z > R$ on the rotating axis.

(b) the magnetic dipole moment.

6. Assume that you have an electric charge e and a magnetic g separated by a distance d . The field of the electric charge is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{e}}{r^2}$$

of course, and the field of the magnetic charge is assumed to be

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{\mathbf{g}}{r^2}$$

Find the total angular momentum stored in the resulting electromagnetic fields. (The above system is known as Thompson's monopole.) Show that your result is independent of the separation distance and points in the direction from e toward g . (Is this a surprising result? Why or why not?) In quantum mechanics angular momentum comes in half-integer multiples of \hbar , so this problem suggests that if magnetic monopoles exist, electric and magnetic charge must be quantized, according to the relation

$$\frac{\mu_0 e g}{4\pi} = \frac{n\hbar}{2}$$

for $n = 1, 2, \dots$, thus explaining this sore point. (This idea was first proposed by Dirac in 1931.)

7. The electromagnetic field tensor (4-D) is defined in SI units by:

$$f_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}, \text{ where } x_\mu = (x, y, z, ict), A_\mu = (A_x, A_y, A_z, i\phi/c)$$

is the 4-vector potential, ϕ is the scalar electrostatic potential, and c is the speed of light. The field tensor may also be written as:

$$f_{\mu\nu} = \begin{pmatrix} 0 & B_z & -B_y & -iE_x/c \\ -B_z & 0 & B_x & -iE_y/c \\ B_y & 0 & 0 & -iE_z/c \\ iE_x/c & iE_y/c & iE_z/c & 0 \end{pmatrix}$$

\vec{E} and \vec{B} are the electric and magnetic fields.

Use the expression $\partial f_{\mu\nu} / \partial x_\nu = \mu_0 J_\mu$ [$J_\mu = (J_x, J_y, J_z, ic\rho)$ is the current four-vector] to obtain two of Maxwell's four equations in SI units. What are the other two? State clearly (1) the name of the physicist whose law is incorporated in each of the 4 equations, and (2) the physical meaning of each of the 4 equations.

BAYLOR

U N I V E R S I T Y

1998 Ph.D. Preliminary Examination

Part IV: General, Mathematical & Statistical Physics

Time: 3 hours

Date: May 29, 1998

Instructions: Work five (5) of the following seven problems. Work each problem on a separate sheet of paper and turn in only the five problems you wish to have graded. Each problem is designed to require approximately 30 minutes to solve. Use your time wisely and be careful to not spend too much time on any one problem. **Good Luck!**

1. Consider the equation

$$\psi''(z) - z\psi'(z) + 2\psi(z) = 0$$

(a) Show that there exists a polynomial $\psi_1(z)$ of the second degree, which satisfies this equation.

(b) Let $\psi_2(z) = \psi_1(z)\chi(z)$ be another solution of the equation, which is linearly independent of the polynomial $\psi_1(z)$. Derive an expression for $\chi(z)$ in the form of an integral and determine the first three terms in the power series expansion of $\psi_2(z)$ about the origin.

2. When a piano string of length L is struck by a piano hammer near the point $x = \xi$, the initial distribution of velocities is given by;

$$v_0(x) = \begin{cases} v_0 \cos \frac{\pi(x-\xi)}{d} & \text{for } |x-\xi| < \frac{d}{2} \\ 0 & \text{otherwise} \end{cases}$$

The initial displacement $u(x,0)$ is zero. Find the solution $u(x,t)$ of the wave equation.

3. Use the Mellin inversion integral to find the following inverse Laplace transforms.

a. $L^{-1} \left\{ \frac{1}{(s^2 + a^2)} \right\}$

b. $L^{-1} \left\{ \frac{1}{\sqrt{s}} \right\}$

4. The rotational motion of diatomic molecules contribute to the specific heat, C_V , in a gas of molecules. In this problem you are to calculate this contribution by considering a set of distinguishable rigid rotators each with moment of inertia I and energy levels

$$E_j = j(j + 1) B, \quad j = 0, 1, 2, \dots$$

where $B = \hbar^2/2I$.

(a) Find the partition function.

(b) Evaluate the partition function for high temperature ($B/kT \ll 1$) by taking $\sum_j \Rightarrow \int dj$. Find the internal energy (U) and specific heat (C_V) at high temperature.

(c) Evaluate the partition function for low temperature ($B/kT \gg 1$) by keeping only the first two terms in the summation. Find U and C_V at low temperature.

5. Consider a system of N distinguishable particles distributed in two non-degenerate energy levels $E_1 = 0$ and $E_2 = \epsilon$ in equilibrium with a reservoir at a temperature T . Calculate (a) the partition function, (b) the fractions N_1/N and N_2/N of particles in each state, (c) the internal energy, U , of the system, and (d) the entropy, S , of the system. What would be the partition function and entropy if the particles were indistinguishable?

6. Electromagnetic radiation in an evacuated vessel of volume V , at equilibrium with the walls at a temperature T , behaves very much like a gas of photons, having an internal energy U and pressure P : $U = aVT^4$, $P = (1/3)aT^4$, where a is Stefan's constant. (This is the starting point for several theorems in the study of classical stellar evolution.)

a. Find the relationship between P and V for an adiabatic process and the relationship between heat absorbed and change of volume in an isothermal process, for this gas.

b. Plot the closed curve, on the P - V plane, for a Carnot cycle.

c. Calculate the heat absorbed and the work done in each part of the cycle and find the efficiency.

7. A slightly non-ideal gas has an equation of state given by

$$P = kT \left[\frac{N}{V} + \left(\frac{N}{V} \right)^2 B(T) \right]$$

where $B(T)$ is small.

a. Given that C_v for an ideal gas is $(3/2)Nk$, find C_v for the above equation of state.

b. Calculate the entropy S . (Hint: Consider the Helmholtz free energy $A = U - TS$ and show that

$$\left(\frac{\partial S}{\partial V} \right)_T = \left(\frac{\partial P}{\partial T} \right)_V$$

Be sure to state clearly any assumptions you find it necessary to make.)

BAYLOR

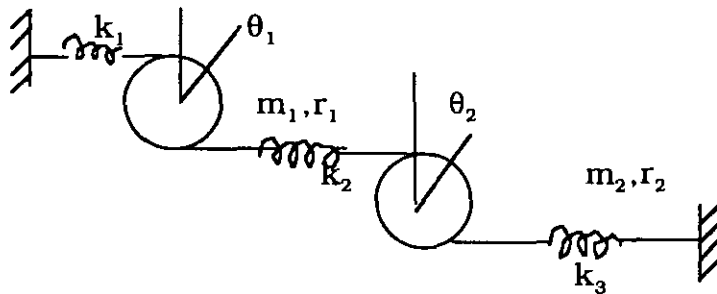
U N I V E R S I T Y

1999 Ph.D. Preliminary Examination Part I: Mechanics

Time: 3 hours
Date: May 27, 1999

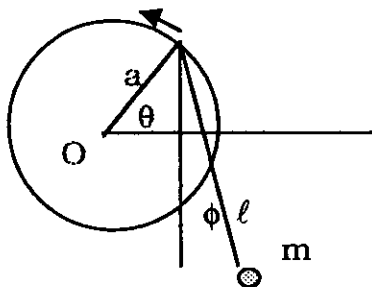
Instructions: Work five (5) of the following six problems. Work each problem on a separate sheet of paper and turn in only the five problems you wish to have graded. Each problem is designed to require approximately 30 minutes to solve. Use your time wisely and be careful to not spend too much time on any one problem. **Good Luck!**

1. (30 minutes)



Two uniform solid cylinders of radii r_1 , r_2 and masses m_1 and m_2 are pivoted about their centers as shown and are connected on tangents to their surfaces by three springs k_1 , k_2 , and k_3 as shown.

- (a) Use the angular variables θ_1 , θ_2 to set up the two equations of motion and from these set up the secular determinant to determine the two eigenfrequencies in terms of the quantities given in the figure. Before solving the secular determinant, however, put $r_1 = r_2 = a$, $m_1 = m_2 = m$, and $k_1 = k_2 = k_3 = k$. Then find the two eigenfrequencies ω_1^2 and ω_2^2 .
- (b) Now find the normalized eigenvectors \vec{a}_1 and \vec{a}_2 and state their physical meaning.

2. (30 min)

A simple pendulum of length ℓ (mass of rod = 0) with mass point m has a radius of suspension that rotates with angular velocity $\dot{\theta}$ on the vertical circle of radius a as shown in the figure to the left. All the motion is confined to a single, fixed vertical plane.

- (a) Set up the Lagrangian in terms of θ , ϕ , $\dot{\theta}$, and $\dot{\phi}$. (Use 0 as the reference point for the potential energy.)
- (b) Use the matrix method to find the Hamiltonian in terms of p_θ , p_ϕ , θ , and ϕ .
- (c) Write down Hamilton's four equations using the Hamiltonian from part (b).

3. (30 min) Find the condition that allows stable circular orbits to exist in a force field described by the "screened Coulomb potential" $U(r)$

$$U(r) = -\frac{k}{r} e^{-(r/a)}$$

where $k > 0$ and $a > 0$.

- 4. (30 min)** (a) Calculate the inertia tensor for a cube with its side of length a . One corner of the cube is at the origin, thus the three adjacent edges lie along the coordinate axes.
- (b) Find the principal moments of inertia.

5. (30 min) Deduce the form of a scattering potential field $U(r)$, given the effective cross section as a function of the angle of scattering for a given energy E . You can assume that $U(r)$ decreases monotonically with r (a repulsive field) and that $U(0) > E$ and $U(\infty) = 0$.

HINT: Define

$$s = 1/r \quad x = 1/\rho^2 \quad \omega = \sqrt{1 - U/E}$$

and then show that

$$\omega = \exp\left(\frac{1}{\pi} \int_{r_0}^{\infty} \cosh^{-1}(\rho/r\omega) (d\chi/d\rho) d\rho\right)$$

(In the above, ρ is the impact parameter and χ is the angle through which the particle is deflected as it passes the center. Be careful not to confuse x with χ in the above!) This formula determines implicitly the function $\omega(r)$ and therefore $U(r)$ for all $r > r_{\min}$.

6. A train at latitude λ in the northern hemisphere is moving due north with a speed v along a straight and level track. Which rail experiences the larger vertical force? Show that the ratio R of the vertical forces on the rails is given by

$$R = 1 + \frac{8\Omega v h \sin \lambda}{ga}$$

where h is the height of the center of gravity of the train above the rails which are at a distance a apart, g is the acceleration due to gravity, and Ω is the angular velocity of the Earth's rotation.

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U N I V E R S I T Y

1999 Ph.D. Preliminary Examination
Part II: Quantum Mechanics

Time: 3 hours
Date: May 27, 1999

Instructions: Work five (5) of the following six problems. Work each problem on a separate sheet of paper and turn in only the five problems you wish to have graded. Each problem is designed to require approximately 30 minutes to solve. Use your time wisely and be careful to not spend too much time on any one problem. **Good Luck!**

1. The Hamiltonian of a particle of mass m interacting with a time independent potential $V(\vec{x})$ is given by

$$H = \frac{\vec{p}^2}{2m} + V(\vec{x}).$$

a) Show that

$$m \frac{d}{dt} \langle \vec{x} \rangle = -\langle \vec{p} \rangle,$$

where $\langle \dots \rangle$ represents expectation value.

b) Now show that

$$\frac{d}{dt} \langle \vec{p} \rangle = -\langle \vec{\nabla} V(\vec{x}) \rangle,$$

where $\vec{\nabla}$ is the gradient operator. [Putting a) and b) together gives

$$m \frac{d^2}{dt^2} \langle \vec{x} \rangle = -\langle \vec{\nabla} V(\vec{x}) \rangle.$$

This is called "Ehrenfest's theorem".]

2. a) Consider three very weakly interacting *distinguishable* spin 1/2 particles of mass m in a one dimensional "box" of length L with one-particle energy levels given by

$$E_n = \frac{h\pi^2 n^2}{2mL^2} \quad (n = 1, 2, 3, \dots)$$

Find the first three energy levels of the the system and the degeneracy of each level.

- b) Now answer the same question in a) but for three *indistinguishable* spin 1/2 particles.

3. The p ($\ell = 1$) states of an electron are described by the orbital functions Y_1^m ($m = 1, 0, -1$), and the spin functions $\chi(+1/2) = \alpha$ and $\chi(-1/2) = \beta$.

- a. With the state of total angular momentum $j = 3/2$, $m_j = 3/2$ given by

$$\left| \frac{3}{2}, \frac{3}{2} \right\rangle = Y_1^1 \alpha$$

use the lowering operator $\mathbf{J} = \mathbf{L} + \mathbf{S}$ to calculate the normalized state $|3/2, 1/2\rangle$ in terms of the orbital and spin functions.

- b. Construct another normalized state with $m = 1/2$ which is orthogonal to the state $|3/2, 1/2\rangle$, and show explicitly that this corresponds to $j = 1/2$.
- c. If $\mathbf{K} = \mathbf{L} + 2\mathbf{S}$, find the expectation values of K_z in the state $|3/2, 1/2\rangle$ and $|1/2, 1/2\rangle$.

4. Consider a pair of identical one-dimensional harmonic oscillators coupled to each other. The Hamiltonian is given by

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}m\omega^2 x_1^2 + \frac{1}{2}m\omega^2 x_2^2 + \frac{1}{2}k(x_1 - x_2)^2$$

Treating the coupling term $\frac{1}{2}k(x_1 - x_2)^2$ as a perturbation, calculate the energy of the ground state in the first order.

5. A spin $1/2$ particle with magnetic moment $\boldsymbol{\mu} = g\mu_B \mathbf{s}$ is at rest in the magnetic field $\mathbf{H} = H \hat{z}$ for time $t < 0$, and for time $t > 0$ in the field $\mathbf{H} = H \hat{x}$.
- Write down the Hamiltonian of the system and find energy eigenvalues and eigenfunctions of the system, both for time $t < 0$ and for time $t > 0$.
 - If the particle is in its highest energy eigenstate for $t < 0$, what is its wavefunction for $t > 0$?
 - Find the expectation values of the components of \mathbf{s} for all times t . If the particle is in its highest energy eigenstate for $t < 0$, what is its wavefunction for $t > 0$?
6. You are given two Hermitian operators H_0 and H' . They have the following properties:
- $[H_0, H'] = 0$
 - $H_0 |u_i\rangle = E_{0i} |u_i\rangle$

where $\{|u_i\rangle\}$ form a complete set and the $\{E_{0i}\}$ are discrete, but not necessarily degenerate.

Show that the eigenfunctions and eigenvalues of $H = H_0 + H'$ are given exactly by the lowest order perturbation theory. (Let $H |w_i\rangle = E_i |w_i\rangle$.)

(Hint: You do not need to explicitly develop the perturbation series to show this result.)

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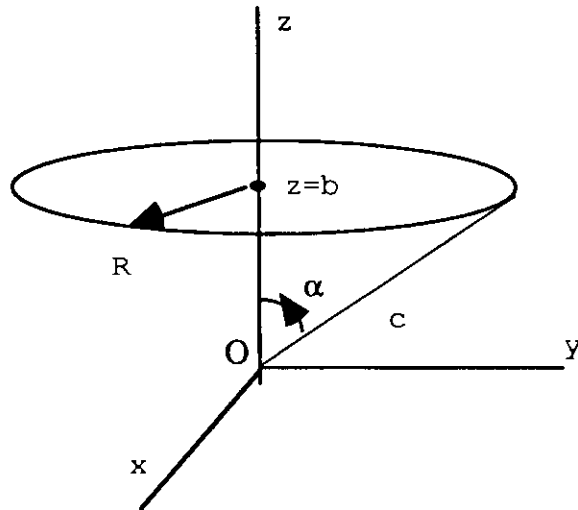
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1999 Ph.D. Preliminary Examination
Part III: E&M

Time: 3 hours
Date: May 28, 1999

Instructions: Work five (5) of the following six problems. Work each problem on a separate sheet of paper and turn in only the five problems you wish to have graded. Each problem is designed to require approximately 30 minutes to solve. Use your time wisely and be careful to not spend too much time on any one problem. **Good Luck!**

1. A total charge q is uniformly distributed around a circular ring of radius R whose plane is perpendicular to the z -axis as shown. Choose an origin such that the center of the ring is located a distance b from the origin, O .



Using the Coulomb expansion,

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{n=0}^{\infty} \frac{r_{<}^n}{r_{>}^{n+1}} P_n(\cos\gamma),$$

where γ is the angle between \vec{x} and \vec{x}' , show that the potential can be written as (cgs units; $c^2 = R^2 + b^2$)

$$\Phi(r, \theta) = q \sum_{n=0}^{\infty} \frac{r_{<}^n}{r_{>}^{n+1}} P_n(\cos\alpha) P_n(\cos\theta),$$

where $r_{<}$ is the lesser of (r, c) and $r_{>}$ is the greater.

2. A closed current loop carrying a current I is bent into an irregular shape. Show that its magnetic moment

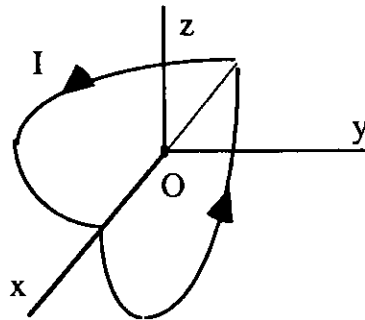
$$\vec{m} = \frac{I}{2c} \oint \vec{r} \times d\vec{l} ,$$

can be written as

$$\vec{m} = \frac{I}{c} \oint da \hat{n} ,$$

where \hat{n} is the appropriately directed surface normal.

b) A circular current loop with radius R is bent 90° along the x -diagonal, as shown.



Find the magnetic moment of the loop from the part a) result above.

3. An infinitely long cylindrical shell of radius R carries a circumferential surface current density $\mathbf{K} = k \hat{\phi}$. Use the Biot-Savart law in the form of

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\mathbf{r}') \times \hat{r}}{r^2} da ,$$

to show the magnetic field (due to \mathbf{K}) at a distance d from the cylinder axis is given by

$$\mathbf{B} = \frac{\mu_0}{2} k \left[\frac{R^2 - d^2}{|R^2 - d^2|} + 1 \right] \hat{z} .$$

4. A beam of light is incident normally from air (refractive index n_1) on a plane slab of a transparent dielectric with refractive index n_2 and thickness h . The light passes through the slab and enters a third medium with refractive index n_3 and infinite extent. Show that the reflection coefficient of the slab is given by

$$R = \frac{r_{12}^2 + r_{23}^2 + 2r_{12}r_{23} \cos 2k_2h}{1 + r_{12}^2 r_{23}^2 + 2r_{12}r_{23} \cos 2k_2h}$$

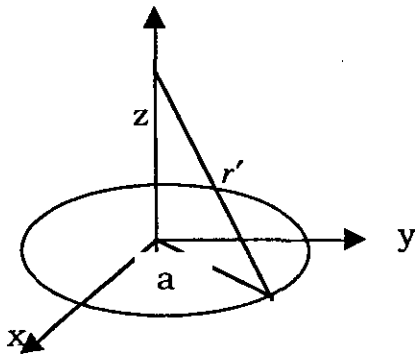
where

$$r_{12} = \frac{n_2 - n_1}{n_2 + n_1}, \quad r_{23} = \frac{n_3 - n_2}{n_3 + n_2}$$

and k_2 is the wave number for the dielectric medium. Assume all the media are nonconducting and nonmagnetic ($\mu = 1$).

(b) Use R to find the condition for zero reflection back into the first medium. This of course is the principle of coated optical lenses.

5.



(a) Use the Biot-Savart Law to show that the z -component, B_z , of the magnetic field at a distance z above the plane of a circular wire (radius a) carrying a current I is given by:

$$B_z = \frac{\mu_0 I a^2}{2(a^2 + z^2)^{3/2}}, \quad (\text{SI units})$$

(b) This magnetic field may be expressed in terms of a magnetic scalar potential

$$\vec{B}(\vec{r}) = -\nabla\Phi_M(\vec{r}) \quad \text{to yield:}$$

$$\Phi_M = -\frac{\mu_0 I a^2}{2} \int \frac{dz}{(a^2 + z^2)^{3/2}} = -\frac{\mu_0 I z}{2(a^2 + z^2)^{1/2}}.$$

For $r > a$, this can be expanded in a power series to get:

$$\Phi_M = -\frac{\mu_0 I}{2} \left(1 - \frac{a^2}{2z^2} + \frac{3a^4}{8z^4} - \frac{5a^6}{16z^6} + \dots \right).$$

(Note: You are given this expression—you don't need to derive it.)

Compare this latter power-series expansion to:

$$\Phi_M = \sum_{\ell=0}^{\infty} \left[A_{\ell} r^{\ell} + \frac{B_{\ell}}{r^{\ell+1}} \right] P_{\ell}(\cos\theta)$$

to obtain the coefficients A_0 (all other A_{ℓ} 's vanish), B_0 , B_1 , B_2 , B_3 , and B_4 . (In other words, the (b) part of this problem is to find these six coefficients in terms of I , a , μ_0 , and thereby determine Φ_M in terms of these coefficients and $P_{\ell}(\cos\theta)$).

(c) From this Φ_M in terms of A_0 , B_0 , B_1 , B_2 , B_3 , and B_4 along with $P_1(\cos\theta) = \cos\theta$ and $P_3(\cos\theta) = (1/2)(5\cos^3\theta - 3\cos\theta)$, obtain

$$B_r = -\frac{\partial\Phi_M}{\partial r}, \quad B_{\theta} = -\frac{1}{r} \frac{\partial\Phi_M}{\partial\theta}, \quad B_{\phi} = -\frac{1}{r\sin\theta} \frac{\partial\Phi_M}{\partial\phi}.$$

6. Assuming that "Coulomb's Law" for magnetic charges (q_m) read

$$\mathbf{F} = \frac{\mu_0}{4\pi} \frac{q_{m1} q_{m2}}{r^2} \hat{\mathbf{r}}$$

work out the force law for a monopole q_m moving with a velocity \mathbf{v} through electric and magnetic fields \mathbf{E} and \mathbf{B} .

BAYLOR

U N I V E R S I T Y

1999 Ph.D. Preliminary Examination

Part IV: General, Mathematical & Statistical Physics

Time: 3 hours

Date: May 28, 1999

Instructions: Work five (5) of the following six problems. Work each problem on a separate sheet of paper and turn in only the five problems you wish to have graded. Each problem is designed to require approximately 30 minutes to solve. Use your time wisely and be careful to not spend too much time on any one problem. **Good Luck!**

Note: The student will need a hand calculator for some of the problems below.

1. (a) For a free particle, show that the single particle density of states in energy is given by

$$D(\epsilon)d\epsilon = (4\pi Vm/h^3)\sqrt{2m\epsilon}d\epsilon .$$

Now consider a gas composed of a fixed number of bosons in a container of volume V at a temperature T . (b) Find the particle density, N/V , as a function of temperature, and (c) the lowest possible temperature for a given particle density, of this boson gas consistent with Bose-Einstein statistics. (d) Find the average energy per particle for N fermions in a container of volume V at $T = 0^\circ \text{K}$.

2. Show that the partition function, Z_N , of an extreme, relativistic gas consisting of N monatomic molecules with an energy-momentum relationship $E = pc$, where c being the speed of light, is given by,

$$Z_N = (1/N!) [8\pi V(k_B T/hc)^3]^N .$$

Study the thermodynamics of this system to show that

$$PV = U/3 \quad \text{and} \quad U/N = 3k_B T.$$

3. Consider the dynamic equilibrium of an *isothermal* atmosphere. Let N be the number of molecules per unit volume and p be the pressure at height z above the surface of the earth for an air molecule of mass m . Use the ideal gas law $pV = nRT = nN_A kT$ (n = number of moles, N_A = Avogadro's number 6.02×10^{23} molecules/gram-mole, $k = 1.38 \times 10^{-23}$ J/K, T the Kelvin temperature) to show that the density ρ of the gas (air) as a function of height z is

$$\rho = \rho_0 e^{-\frac{mgz}{kT}}$$

- (b) (Numerical calculation) Show that this density satisfies the condition of hydrostatic equilibrium, namely, that the height of the atmosphere is equivalent to the height of a column of water 10.3 meters high (or a column of Hg 0.76 m high). You are to determine the height of the atmosphere and to discuss whether this height is reasonable or not. Use $T = 20^\circ\text{C}$, $\rho_{\text{H}_2\text{O}} = 1.00$ gm/cm³, $\rho_{\text{Hg}} = 13.6$ gm/cm³, $\rho_{\text{air}} = 1.29$ kg/m³, and $m_{\text{H}} = 1.67 \times 10^{-27}$ kg, $g = 9.80$ m/s².
4. The equation of motion for the amplitude of vibration of a string with fixed end points ($0 \leq x \leq L$) is:

$$\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 y(x,t)}{\partial t^2} = 0$$

- (a) Using the method of separation of variables show that if

$$y(x,t) = u(x) v(t) ,$$

then

$$\frac{d^2 u(x)}{dx^2} + k^2 u(x) = 0 ,$$

and

$$\frac{d^2 v(t)}{dt^2} + \omega^2 v(t) = 0 ,$$

where $\omega = ck$.

4b. Find the normal modes of vibration by solving these eigenvalue problems. Give the general solution for an arbitrary initial state ($t = 0$) of vibration.

(c) Compute the Green's function $G(x, x'; k)$ for the operator $d^2/dx^2 + k^2$ in closed form by solving the equation

$$\frac{d^2 G(x, x'; k)}{dx^2} + k^2 G(x, x'; k) = \delta(x - x')$$

subject to the given boundary conditions.

- 5.** Use the virial theorem to derive the ideal gas law.
- b.** Carry out an expansion of this study to include a classical system of particles interacting through a two-body potential $u(\mathbf{r}_j - \mathbf{r}_i)$. (You may assume that this interparticle potential is central and denote it as $u(r)$ where $r = |\mathbf{r}_j - \mathbf{r}_i|$.) Show that

$$PV = NkT \left[1 - \frac{2\pi n}{3kT} \int_0^\infty \frac{\partial u(r)}{\partial r} g(r) r^3 dr \right]$$

where n is the particle density of the system and $g(r)$ is the pair distribution function or the measure of probability of finding a pair of particles in the system separated by a distance r . (In this case, $g(r) \rightarrow 1$ as $r \rightarrow \infty$.)

- c.** Show that the internal energy of the system can also be expressed in terms of $u(r)$ and $g(r)$ as

$$U = \frac{3}{2} NkT \left[1 + \frac{4\pi n}{3kT} \int_0^\infty u(r) g(r) r^2 dr \right]$$

- 6.** Consider a system of $N \gg 1$ non-interacting particles in which the energy of each particle can assume two and only two distinct values: 0 and E ($E > 0$). Denote by n_0 and n_1 the occupation numbers of the energy levels 0 and E , respectively. The fixed total energy of the system is U .
- a.** Find the entropy of the system.
- b.** Find the temperature as a function of U . For what range of values of n_0 is $T < 0$?
- c.** In which direction does heat flow when a system of negative temperature is brought into thermal contact with a system of positive temperature? Why?

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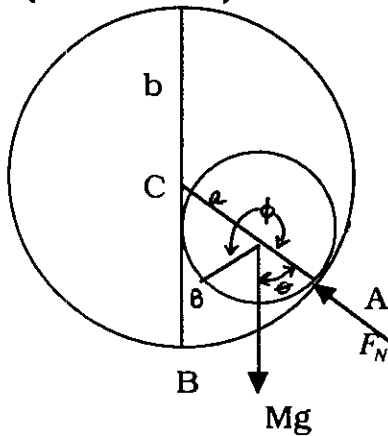
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2000 Ph.D. Preliminary Examination
Part I: Mechanics

Time: 3 hours
Date: May 25, 2000

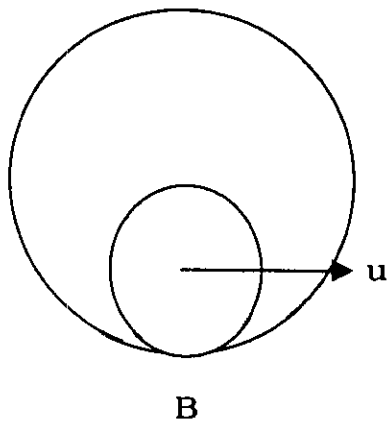
Instructions: Work five (5) of the following six problems. Work each problem on a separate sheet of paper and turn in only the five problems you wish to have graded. Each problem is designed to require approximately 30 minutes to solve. Use your time wisely and be careful to not spend too much time on any one problem. **Good Luck!**

1. (35 minutes)



A uniform solid sphere of radius a , $I = \frac{2}{5}Ma^2$, rolls without slipping on the inside of a rough cylinder of radius b whose axis is horizontal. The sphere is started rolling at the lowest point B with a horizontal velocity \underline{u} .

(a) Use $b\theta = a(\phi + \theta)$ to set up the Lagrangian and find the Euler-Lagrange equation for the variable θ , where θ is the angle between CB and CA , and ϕ is the angle subtended at the center of the sphere by arc length AB on the sphere.



b) Find the minimum value for \underline{u} for which the sphere will roll around the cylinder without leaving the cylinder at the top.

Ans. $u^2 \geq \frac{27}{7}(b-a)g$.

2. (30 min) A particle of mass m moves in attractive central force field defined by $\vec{F} = -k\vec{e}_r/r^3$, where \vec{e}_r is a unit vector in the r -direction. The particle starts on the positive x -axis at a distance a away from the origin and moves with speed v_0 in a direction making an angle α with the positive x -axis.

(a) Prove that the differential equation for the motion is given by

$$\frac{d^2 r}{dt^2} = -\frac{(k - ma^2 v_0^2 \sin^2 \alpha)}{mr^3}.$$

(b) Show that the above differential equation can be written in terms of $u = 1/r$ as

$$\frac{d^2 u}{d\theta^2} + (1 - \gamma)u = 0, \text{ where } \gamma = \frac{k}{ma^2 v_0^2 \sin^2 \alpha}.$$

(c) Solve the differential equation in part (b) and interpret the result physically. Be sure to give a meaningful discussion, and not something trivial. Credit will be lost for a trivial or meaningless discussion.

3. (30 min) Let $\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be an eigenvector of the real symmetric matrix

$$\mathbf{A} = \begin{bmatrix} -9 & -3 & 0 \\ -3 & 12 & -3 \\ 0 & -3 & 9 \end{bmatrix}$$

- a. Write down the algebraic equation for x, y, z corresponding to the eigenvalue equation $\mathbf{A}\mathbf{u} = \lambda\mathbf{u}$.
- b. Consider the quadratic form associated with the matrix \mathbf{A} , namely,

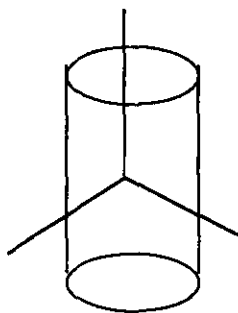
$$F(x, y, z) = 9x^2 + 12y^2 + 9z^2 - 6xy - 6yz.$$

- Show that the problem of finding an extremum of $F(x, y, z)$, subject to the constraint $G(x, y, z) = x^2 + y^2 + z^2 = \text{constant}$, leads to the equations found in part (a) when treated with Lagrange multipliers. Also, show that essentially the same equations are obtained when we look for an extremum of $G(x, y, z)$ subject to $F(x, y, z) = \text{constant}$.
- c. Show that the points on the quadratic surface
- $$9x^2 + 12y^2 + 9z^2 - 6xy - 6yz = 18,$$
- which make the distance from the origin stationary, can be found by solving problem (b). Find these points, show that the given quadratic is an ellipsoid, and find its principal axes.
- d. What is the relationship between the eigenvalues and eigenvectors of the matrix \mathbf{A} and the principal axes of the ellipsoid?

4. (30 min) Find all the normal modes and the corresponding frequencies of the two-dimensional system shown below. Three rigid spheres are connected by massless, flexible rods with $m_1 = m$, $m_2 = 2m$, $m_3 = m$ (from left to right).



5. (30 min) Consider a particle of mass m that is constrained to move on the surface of a cylinder shown below. The radius of the cylinder is R , and the long symmetric axis is along the z direction. If the particle is subject to a force directed toward the origin such as $\mathbf{F} = -k\mathbf{r}$ (\mathbf{r} is the position vector of the particle from the origin), first formulate this problem by writing the Lagrangian. Then show that the angular momentum about the z -axis is a constant of the motion and the motion along the z -axis is simple harmonic.



6. (30 min) Consider the simple pendulum, a mass m , constrained by a wire of length ℓ to swing in an arc. Using the generalized coordinates r and θ (motion in the vertical plane) find the Lagrangian. Take V to be zero when the pendulum is horizontal. (b) Find the equations of motion in terms of a Lagrange multiplier λ_1 . (c) State the equation of constraint. (d) Use the equation of constraint along with the assumption that the amplitude is small to solve for $\theta(t)$. (e) Find λ_1 in terms of θ and the first derivative of θ with respect to time. What does this represent physically?

BAYLOR

U N I V E R S I T Y

2000 Ph.D. Preliminary Examination
Part II: Quantum Mechanics

Time: 3 hours
Date: May 25, 2000

Instructions: Work five (5) of the following six problems. Work each problem on a separate sheet of paper and turn in only the five problems you wish to have graded. Each problem is designed to require approximately 30 minutes to solve. Use your time wisely and be careful to not spend too much time on any one problem. **Good Luck!**

1. (30 minutes) A normalized harmonic oscillator coherent-state wave function is given by

$$|\lambda\rangle = \exp\left(-\frac{|\lambda|^2}{2}\right) \exp(\lambda a^\dagger) |0\rangle,$$

$$[a, a^\dagger] = 1,$$

where $|0\rangle$ is the oscillator ground state energy eigenket, λ is a complex number, and a^\dagger and a are creation and annihilation operators, respectively.

(a) Show that

$$a|\lambda\rangle = \lambda|\lambda\rangle.$$

(b) Now show

$$\frac{\Delta N}{\langle \lambda | N | \lambda \rangle} = \frac{1}{\sqrt{\langle \lambda | N | \lambda \rangle}},$$

where ΔN is the uncertainty in the value of the number operator, $N = a^\dagger a$, and $\langle \lambda | N | \lambda \rangle$ is its expectation value.

2. (30 minutes) An integral equation for bound states for the Schrödinger equation may be derived as follows. Start with the formal equation

$$|\psi\rangle = \frac{1}{E-H_0} V|\psi\rangle,$$

where E is a negative number (set $E = -\frac{\hbar^2\kappa^2}{2m}$, $\kappa > 0$) and V is the local potential. To keep things simple, imagine we are in one spatial dimension.

(a) By inserting a complete set of states, show that the above equation leads to the bound state integral equation,

$$\langle x|\psi\rangle = -\frac{m}{\hbar^2\kappa} \int_{-a}^a dx' e^{-\kappa|x-x'|} V(x') \langle x'|\psi\rangle,$$

for a potential which is bounded by $-a < x' < a$. (Alternate means of proof: show by substitution that the above exactly solves the one-dimensional Schrödinger equation.)

(b) Using the result of (a), solve for the bound state energy and wavefunction of a potential given by a delta function,

$$V(x) = -\frac{\gamma\hbar^2}{2m} \delta(x), \quad \gamma > 0.$$

3. (30 minutes) In the ground state of a hydrogen atom, the electron is subjected at $t = 0$ to a time-dependent perturbation of the form $H' = g\cos\theta\exp(-\gamma t)$ with $\gamma > 0$ where r and θ are the polar coordinates of the electron and g is a small parameter.

- Compute the transition probability for the electron to go into an excited state $|\phi\rangle$ at $t = \infty$, using the lowest order perturbation theory.
- Obtain the angular momentum selection rules of the transition.

4. (30 minutes) A particle of mass m moves in a potential $V(r) = -V_0$ when $r < a$, and $V(r) = 0$ when $r > a$. Find the least value of V_0 such that there is a bound state of zero energy and zero angular momentum.

5. (30 minutes) A particle of mass m is placed in a finite spherical potential

$$V(r) = \begin{cases} 0 & r \leq a \\ V_0 & r > a \end{cases}$$

a. What minimum value must V_0 have in order for a bound state to exist?

b. Let $V_0 = \frac{\pi^2 \hbar^2 a^2}{4m}$. Find the ground state energy by solving the radial equation for $\ell = 0$. (Note: You will have to obtain the solution numerically, since there is not a closed form solution.)

6. (30 minutes) Write down the Schrödinger equation for a Helium atom.

b. If the electron-electron interaction is neglected, use your knowledge of the ground state energy of hydrogen (i.e. $E_0 = -\frac{m}{2\hbar^2} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 = -13.6$ eV), to write down the approximate ground state energy for He.

c. Calculate $\left\langle \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} \right\rangle$ for the state

$$y_0(\mathbf{r}_1, \mathbf{r}_2) = y_{100}(\mathbf{r}_1) y_{100}(\mathbf{r}_2) = \frac{8}{\pi a^3} e^{-2(r_1+r_2)/a}.$$

(Hint: Your answer should be a^{-1} times a rational number.)

d. Use your result from part (c) to estimate the ground state energy of helium when the electron-electron interaction is included. Express your answer in electron-volts. (Hint: The experimental value is -79 eV.)

BAYLOR

U N I V E R S I T Y

2000 Ph.D. Preliminary Examination Part III: E&M

Time: 3 hours
Date: May 26, 2000

Instructions: Work five (5) of the following six problems. Work each problem on a separate sheet of paper and turn in only the five problems you wish to have graded. Each problem is designed to require approximately 30 minutes to solve. Use your time wisely and be careful to not spend too much time on any one problem. **Good Luck!**

1. (30 min.) Imagine that new precise measurements have revealed an error in Coulomb's law. The actual force of interaction between two point charges is found to be

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \left(1 + \frac{r}{\lambda}\right) e^{-r/\lambda} \hat{r} \quad ,$$

where λ is a new constant (a huge number with dimensions of length). You are charged with the task of reformulating electrostatics to accommodate the new discovery. Assume the principle of superposition still holds.

(a) What is the electric field of a charge distribution with charge density ρ ?

(b) Show this electric field admits a scalar potential.

(c) Find the potential of a point charge q using infinity as your zero potential reference point.

(d) For a point charge q at the origin, show that

$$\oint \vec{E} \cdot d\vec{a} + \frac{1}{\lambda^2} \int_V \nabla^2 \tau = \frac{1}{\epsilon_0} q$$

for a spherical surface.

(e) Show that the result in (d) holds for a non-spherical surface.

(f) Find an equation to replace the Poisson's equation for the new law.

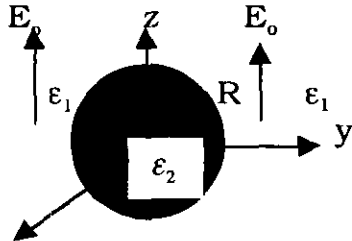
2. (30 min.) A very long solenoid of radius a , with n turns per unit length, carries a current I_S . Coaxial with the solenoid, at radius $b \gg a$, is a circular ring of wire, with resistance R . When the current in the solenoid is (gradually) decreased, a current I_R is induced in the ring.

(a) Calculate I_R , in terms of dI_S/dt .

(b) The power ($I_R R^2$) delivered to the ring must come from the solenoid.

Confirm this by calculating the Poynting vector just outside the solenoid. (The electric field is due to the changing flux in the solenoid; the magnetic field is due to the current in the ring.) Integrate over the entire surface of the solenoid taking the solenoid as infinitely long, and prove that you recover the correct total power $I_R R^2$.

3. (30 minutes)



A dielectric sphere (dielectric constant ϵ_2) of radius R is placed in a medium of dielectric constant ϵ_1 and uniform electric field E_0 parallel to the z -axis as shown.

The solution to Laplace's equation $\nabla^2\Phi = 0$ in spherical coordinates for the regions inside and outside the sphere is

$$\Phi_1(r, \theta) = -E_0 r \cos\theta + \sum_{\ell=1}^{\infty} A_{\ell} \frac{P_{\ell}(\cos\theta)}{r^{\ell+1}}, \quad r > R, \text{ and}$$

$$\Phi_2(r, \theta) = \sum_{\ell=0}^{\infty} B_{\ell} r^{\ell} P_{\ell}(\cos\theta), \quad r < R,$$

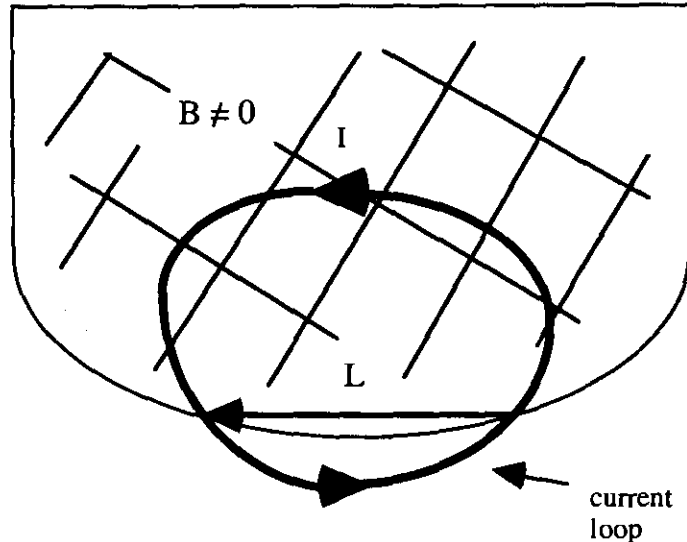
x

where $P_{\ell}(\cos\theta)$ are the well-known Legendre polynomials. Use these two expressions along with the appropriate boundary conditions between the two regions (inside and outside the sphere) to show that the electric field inside the sphere is $\frac{2\epsilon_1 E_0}{\epsilon_2 + 2\epsilon_1}$ and E_0 plus that of a pure dipole moment $\frac{R^3 E_0 (\epsilon_2 - \epsilon_1)}{\epsilon_2 + 2\epsilon_1}$ outside the sphere.

(Hint: You will need the well-known relation $\int_{-1}^{+1} P_{\ell}(\mu) P_n(\mu) d\mu = \frac{2}{2\ell+1} \delta_{\ell n}$. Prove that one boundary condition yields $B_0 = 0$, $B_1 R = -E_0 R + A_1/R^2$, $B_2 R^2 = A_2/R^3$, etc. The other boundary condition will give parameters from which the needed coefficients in the expansion are obtained).

4. (30 minutes) Prove a simplified Thompson's theorem for a single closed surface: If a surface is fixed in position and a given total charge is placed on it, then the electrostatic energy in the region bounded by the surface and infinity is an absolute minimum when the charges are placed so that the surface is an equipotential.

5. (30 minutes) (a) An irregularly shaped planar circuit carrying current I is partly immersed in a uniform magnetic field, \vec{B} , as shown. The \vec{B} field points perpendicularly to the plane of the circuit and is nonzero only in the crosshatched region shown.

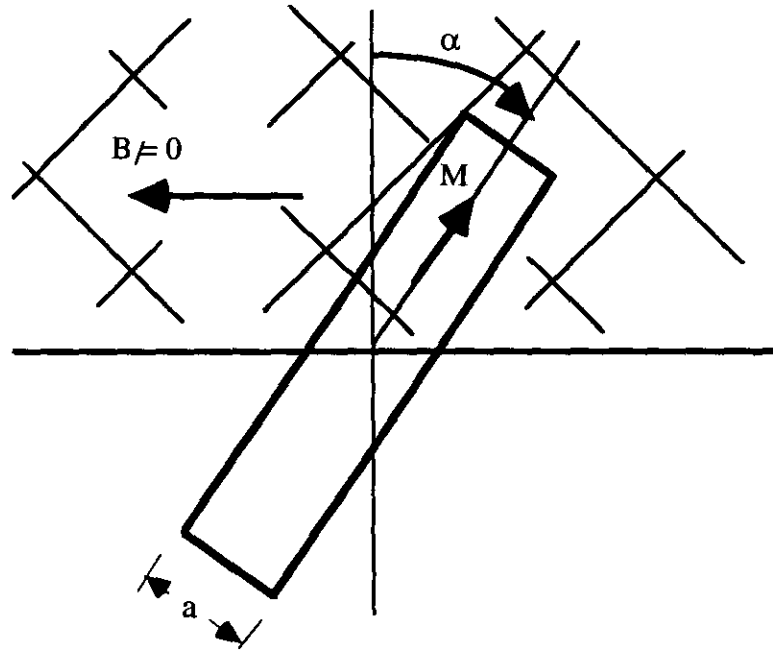


Show that the force on the circuit is given by (cgs units)

$$\vec{F} = \frac{I}{c} \vec{L} \times \vec{B},$$

where \vec{L} is the vector pointing from one side of the circuit to the other where the magnetic field vanishes.

(b) A magnet with uniform volume magnetization \vec{M} down its length and a square cross section (side "a") sticks out of a region of uniform \vec{B} (crosshatched region). It is tipped at an angle α with respect to the perpendicular of the plane defining the $\vec{B} \neq 0$ region. It is oriented such that two of its sides are in the plane defined by \vec{B} and \vec{M} .



Find the force on the rod. (Carefully define your symbols.)

6. (30 minutes) Suppose $V = 0$ and $\mathbf{A} = A_0 \sin(\kappa x - \omega t)\mathbf{j}$, where A_0 , ω and κ are constants. Find \mathbf{E} and \mathbf{B} and then check that they satisfy Maxwell's equations in a vacuum. What condition must you impose on ω and κ ?

BAYLOR

UNIVERSITY

2000 Ph.D. Preliminary Examination

Part IV: General, Mathematical & Statistical Physics

Time: 3 hours

Date: May 26, 2000

Instructions: Work five (5) of the following six problems. Work each problem on a separate sheet of paper and turn in only the five problems you wish to have graded. Each problem is designed to require approximately 30 minutes to solve. Use your time wisely and be careful to not spend too much time on any one problem. **Good Luck!**

Note: The student will need a hand calculator for some of the problems below.

1. (30 min.) A system consists of a volume V containing a variable number of non-conserved particles of mass m . A particle may be created by an expenditure of energy equal to its rest mass energy (mc^2). The particles act like members of an ideal gas. The allowed energy of each particle is mc^2 plus its kinetic energy.

(a) Find the partition function of this non-conserved particle system.

(b) Find the entropy of the system and show its internal energy is given by

$$U = \left(\frac{3}{2} + \frac{mc^2}{k_B T} \right) PV.$$

2. (30 min.) Two identical particles of mass m interact with an external potential which is harmonic, but they do not interact with each other. The Hamiltonian of the system is

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{m\omega^2}{2} (x_1^2 + x_2^2).$$

What are the energy levels of the system? Given that the particles are bosons, determine the partition function. Repeat the calculation given that the two particles are fermions.

- 3. (30 minutes)** In the atomic spectrum of sodium ($Z = 11$), the spectral line D_1 generated by the transition from the first excited state ${}^2P_{1/2}$ to the ground state ${}^2S_{1/2}$ has a wavelength 5897.6 \AA ($1 \text{ \AA} = 10^{-10} \text{ m}$). When the sodium atom is placed in a magnetic field, the spectral line splits into four spectral lines. Calculate the wavelength shift, relative to the undisturbed line, for each member of this quadruplet, if the magnetic field is 0.2 T . Express your answer in \AA .

(Hint: the Landé g-factor is $g = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}$;

$e = 1.602 \times 10^{-19} \text{ Coul}$, $\hbar = 1.054 \times 10^{-34} \text{ J s}$, $E = hc/\lambda$,

$E' = \frac{hc}{\lambda + \Delta\lambda} = \frac{hc}{\lambda(1 + \Delta\lambda/\lambda)} = \frac{hc}{\lambda} \left(1 - \frac{\Delta\lambda}{\lambda}\right) = E + \Delta E$, $m = 9.109 \times 10^{-31} \text{ kg}$,

and a Bohr magneton is $\frac{e\hbar}{2m}$. **Partial answer: + 0.0216 \AA.**

- 4. (30 minutes)** Consider the differential equation:

$$(1 - x^2)y'' - xy' + \ell y = 0.$$

- Identify the location and type of the equation's singular points.
- Solve the equation with a series expansion. Deduce the values of ℓ which make one solution analytic at the singular points. Show that this solution is a polynomial.
- Develop explicitly several such polynomials, standardizing them by the condition $y(0) = 1$.

- 5. (30 minutes)** Consider a gas contained in a volume V at temperature T . The gas is composed of N distinguishable particles of zero rest mass so that the energy E and momentum \mathbf{p} of the particles are related by $E = pc$. The number of single-particle energy states in the range p to $p + dp$ is

$$\frac{4\pi V p^2 dp}{h^3}$$

- Find the equation of state of the system.

b. Find the internal energy of the gas and compare it to an ordinary gas.

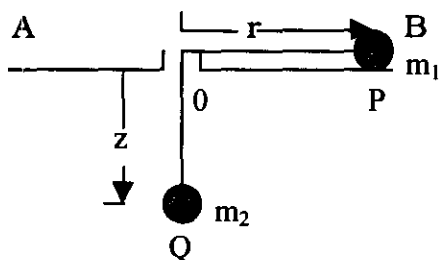
6. (30 minutes) An assembly of N particles of spin $1/2$ are lined up on a straight line. Only nearest neighbors interact. When the spins of the neighbors are both up or both down, their interaction energy is J . When one is up and one is down, their interaction energy is $-J$. (In quantum mechanical language, the energy is $J \sigma_z^i \sigma_z^j$ between a neighboring pair of particles i and j .) What is the partition function of the assembly at a temperature T ? (A consideration of this problem led to the Ising model.)

2001 Ph.D. Preliminary Examination
Part I: Mechanics

Time: 3 hours
Date: May 24, 2001

Instructions: Work five (5) of the following six problems. Work each problem on a separate sheet of paper and turn in only the five problems you wish to have graded. Each problem is designed to require approximately 30 minutes to solve. Use your time wisely and be careful to not spend too much time on any one problem. **Good Luck!**

1. (35 minutes)



AB represents a frictionless horizontal plane having a small opening at O. A string of length ℓ which passes through O has at its ends a particle of mass m_1 at P and a second particle of mass m_2 which hangs freely at Q. The particle at P is given an initial velocity v_0 at right angles to string OP when the length $OP = a$. Let r, θ be the variables in the plane and z be the vertical coordinate shown in the figure.

- Set up the Lagrangian and obtain the three equations using Lagrange undetermined multipliers. Determine the value(s) of the undetermined multiplier(s) and give its (their) physical significance.
- Obtain the differential equation for the motion in terms of r (let $\ell = m_1 a v_0 = m_1 r^2 \dot{\theta}$).
- Find the value of \dot{r} as a function of the initial conditions.

2. (35 minutes) A projectile is fired at an angle of 45° with initial kinetic energy E_0 . At the top of its trajectory, the projectile explodes with additional energy E_0 into two fragments. One fragment of mass m_1 travels straight down. Show that the magnitude of the velocity of the second fragment of mass m_2 is given by

$$v_2 = \sqrt{\frac{E_0(4m_1 + m_2)}{m_2(m_1 + m_2)}}$$

Find the direction of the velocity of m_2 and the velocity of m_1 . What is the ratio of m_1/m_2 when m_1 is a maximum?

3. (35 minutes) The surface of a sphere is vibrating slowly in such a way that the principal moments of inertia are described as

$$I_{zz} = \frac{2mr^2}{5}(1 + \varepsilon \cos \omega t), \quad I_{xx} = I_{yy} = \frac{2mr^2}{5}\left(1 - \varepsilon \frac{\cos \omega t}{2}\right)$$

with $\varepsilon \ll 1$. The sphere is simultaneously rotating with angular velocity $\boldsymbol{\Omega}(t)$. Show that the z-component of $\boldsymbol{\Omega}$ remains approximately constant. Also show that $\boldsymbol{\Omega}(t)$ undergoes the precession around z with the frequency $\omega_p = (3\varepsilon\Omega_z/2) \cos \omega t$ provided $\Omega_z \gg \omega$.

4. (35 minutes) A particle of mass m moves under the influence of gravity on the inner surface of the paraboloid of revolution $x^2 + y^2 = az$ which is assumed to be frictionless. (a) Find the Lagrangian for this system. (b) Find the constraint condition and show that it leads to $2\rho \delta\rho - a \delta z = 0$. (c) Find the equations of motion. (d) Prove that the particle will describe a horizontal circle in the plane $z = h$ provided that it is given an angular velocity whose magnitude is $\omega = \sqrt{2g/a}$. (e) If the particle is displaced slightly from this circular path it will undergo oscillations about the path. What is the frequency of these oscillations?

5. (35 minutes) A disk-shaped body of mass M and moment of inertia I is provided with a deep groove in the median plane perpendicular to its axis. A string is wound on the shaft of radius r in the groove. (This is a yo-yo.) The loose end of the string is held in the hand. One then lets the body fall, with the string taut at all times. As the body descends, it acquires a rotational acceleration until the string is unwound. Then it transitions to a stage where the body shifts from one side of the string to the other. The string now winds around the shaft in the opposite sense, and the body climbs with rotational deceleration. What is the string tension, T :

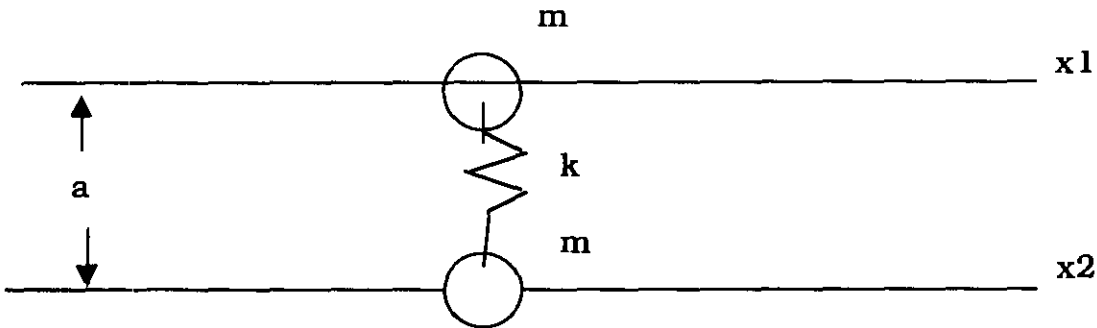
- (a) in descent?
- (b) in ascents?

(Assume that r is so small compared to the distance the body falls that the string is always considered to be vertical.)

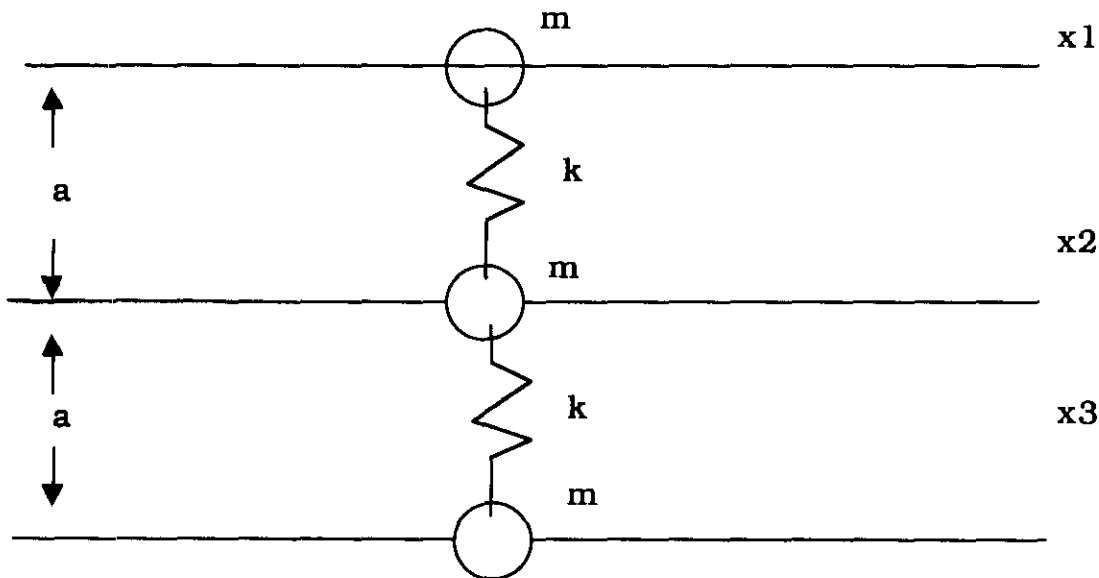
(c) A small point mass, m , moves on the outside of the upper half of a sphere. Let its initial position \mathbf{z}_0 and initial velocity \mathbf{v}_0 be arbitrary, except that the latter is to be tangent to the surface of the sphere. The motion is to be frictionless, solely under the influence of gravity. At what height does the mass leave the sphere?

6. (35 minutes) Consider the system in equilibrium below. The masses are constrained to horizontal motion. The vertical separation of the masses is a . The equilibrium length of the spring is $b < a$. The spring constant is k . Solve the following:

- A. Determine the potential energy of the system as a function of x_1 and x_2 .
- B. What is the effective spring constant for small vibrations ($|x_2 - x_1| \ll a$)? What is the physical significance when $a = b$?
- C. How would the equilibrium points of the system change if $b > a$?



D. For small vibrations, $|x_2 - x_1| \ll a$ and $|x_2 - x_3| \ll a$, determine the normal modes and frequencies for the system below



2001 Ph.D. Preliminary Examination
Part II: Quantum Mechanics

Time: 3 hours
Date: May 24, 2001

Instructions: Work five (5) of the following six problems. Work each problem on a separate sheet of paper and turn in only the five problems you wish to have graded. Each problem is designed to require approximately 30 minutes to solve. Use your time wisely and be careful to not spend too much time on any one problem. **Good Luck!**

1. (30 minutes) For an electron whose spin precesses about a magnetic field \mathbf{B} due to its magnetic moment μ_s , the Hamiltonian corresponding to the spin-energy can be written as:

$$H_{op} = -\omega S_z, \text{ where } \omega = \frac{eB}{m} \text{ is the precessional frequency.}$$

In a particular representation, the three spin operators S_x , S_y , and S_z can be written as:

$$S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, S_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, S_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Use the unitary operator $T_{op} = \exp(i H_{op} t / \hbar)$ to transform the Schrödinger operator S_x to a Heisenberg representation $S_x^{Heisenberg}$. Express your answer as a 3×3 matrix.

2. (30 minutes) Use the variational principle to show that an attractive square-well potential in one dimension, no matter how weak, produces at least one bound state. Take the trial wavefunction to be $\exp(-a^2 x^2)$.

3. (30 minutes) (a) The eigenvalue equation for a system with a discrete, nondegenerate and complete set of states may be written formally as (Dirac notation; assume $\langle \psi_n | \psi_n \rangle = 1$)

$$H_0 | \psi_n \rangle = E_n | \psi_n \rangle,$$

where H_0 is the unperturbed Hamiltonian and the old eigenenergies are given by E_n . A small potential, ΔV , is added to H_0 , $H = H_0 + \Delta V$. Show that the new ground state eigenvector, $|\psi\rangle$, is given approximately by

$$|\psi\rangle \approx |\psi_0\rangle + \sum_{n=1} |\psi_n\rangle \frac{\langle \psi_n | \Delta V | \psi_0 \rangle}{(E_0 - E_n)}.$$

(b) Now show that the approximate shift in the ground state energy is given by

$$\Delta E \approx \langle \psi_0 | \Delta V | \psi_0 \rangle.$$

4. (30 minutes) Consider the quantum mechanical oscillator in one dimension. Using operator techniques express the \mathbf{x} and \mathbf{p} operators in terms of “raising” and “lowering” operators. Express the Hamiltonian in terms of these operators. What is the commutation relation between \mathbf{x} and \mathbf{p} ? With this information find (i.e. derive):

- (a) E_0
- (b) E_n
- (c) U_0
- (d) U_n in terms of U_0 .

5. (30 minutes) Write down the normalized wavefunctions of a system of three identical bosons, which are in given one-particle states.

(b) Now consider three particles located at the corners of an equilateral triangle. Each particle carries a quantum-mechanical spin of $(1/2)$ and their mutual spin Hamiltonian is given by

$$H = (\lambda/3) (\sigma_1 \cdot \sigma_2 + \sigma_1 \cdot \sigma_3 + \sigma_2 \cdot \sigma_3)$$

List the energy levels of this spin system, giving their total spin values and degeneracies. What is the partition function, Z ?

6. (30 minutes) An atom of atomic number Z has only one electron. Assume that the positive charge inside the nucleus is uniformly distributed throughout a sphere of radius R .

a. Calculate the perturbing Hamiltonian to the hydrogen-like Schrödinger equation for zero nuclear size.

b. Calculate the first order correction to the ground state energy.

(Hint: The unperturbed $1s$ wavefunction is $\frac{Z^{3/2}}{\pi^{1/2} a^{3/2}} e^{-Zr/a}$ where a is the hydrogen Bohr radius = $.5292 \times 10^{-10}$ m.)

c. Assuming $R = r_0 A^{1/3}$ where $r_0 = 1.2$ fm, calculate the energy shift for the $1s$ state in ${}^7\text{Li}$.

2001 Ph.D. Preliminary Examination
Part III: E&M

Time: 3 hours
Date: May 25, 2001

Instructions: Work five (5) of the following six problems. Work each problem on a separate sheet of paper and turn in only the five problems you wish to have graded. Each problem is designed to require approximately 30 minutes to solve. Use your time wisely and be careful to not spend too much time on any one problem. **Good Luck!**

1. (30 minutes) A uniform line charge λ is placed on an infinite straight wire, a distance d above a grounded conducting plane. (Assume the conducting plane as xy plane and the wire runs parallel to the x axis at a distance d directly above it).

(a) Show that the potential in the region above the plane is given by

$$V(y,z) = \frac{\lambda}{4\pi\epsilon_0} \ln \left[\frac{y^2 + (z+d)^2}{y^2 + (z-d)^2} \right].$$

(b) Find the induced charge density on the conducting plane.

2. (30 minutes) A localized distribution of charge has a charge density

$$\rho(\vec{r}) = \frac{1}{64\pi r^2} e^{-r} \sin^2 \theta$$

Make a multipole expansion of the potential due to this charge density and determine all the non-vanishing multipole moments. Write down the potential at large distances as a finite expansion in Legendre polynomials.

3. (30 minutes) A sphere of radius 'a' carrying a uniform surface-charge density, σ , is rotating about a diameter with a constant angular velocity, ω . Calculate the magnitude and direction of the magnetic intensity, \mathbf{H} , at the center of the sphere.

It turns out that this value for \mathbf{H} is the same for all points inside the sphere. Describe why this result is reasonable.

4. (30 minutes) A semi-infinite planar dielectric slab, with dielectric constant $\epsilon \neq 1$, is placed parallel to and a distance d above the surface of a perfectly conducting plane. Take the surface of the conducting plane to be $z = 0$ and the surface of the dielectric slab to be at $z = d > 0$. The Green function, $G_D(\mathbf{x}, \mathbf{x}')$, for a positive unit charge in the region between the plate and slab will satisfy the differential equation,

$$-\nabla^2 G_D(\mathbf{x}, \mathbf{x}') = 4\pi \delta(\mathbf{x}, \mathbf{x}'),$$

where $\delta(\mathbf{x}, \mathbf{x}')$ is a Dirac delta function. Given the Bessel function expansions,

$$\delta(\mathbf{x} - \mathbf{x}') = \sum_{m=-\infty}^{\infty} \frac{e^{im(\phi-\phi')}}{2\pi} \int_0^{\infty} dk k J_m(k\rho) J_m(k\rho') \delta(z - z'),$$

$$G_D(\mathbf{x}, \mathbf{x}') = 4\pi \sum_{m=-\infty}^{\infty} \frac{e^{im(\phi-\phi')}}{2\pi} \int_0^{\infty} dk k J_m(k\rho) J_m(k\rho') g_m(z, z'),$$

show that, when boundary conditions at $z = 0, d$ are supplied, this gives

$$g_m(z, z') = \begin{cases} f(z_>) \sinh(kz_<) & , z < d \\ K(z') e^{k(2d-z)} & , z > d \end{cases}$$

where $z_<(>)$ is the lesser (greater) of z and z' and

$$f(z_>) = \frac{e^{kz_>}}{k} \left(\frac{1 + \left(\frac{1+\epsilon}{1-\epsilon}\right) e^{k(2d-z_>)}}{1 + \left(\frac{1+\epsilon}{1-\epsilon}\right) e^{2kd}} \right),$$

$$K(z') = \left(\frac{2}{1-\epsilon} \right) \frac{\sinh(kz')}{k \left(1 + \left(\frac{1+\epsilon}{1-\epsilon}\right) e^{2kd} \right)}.$$

[Note that the radial Laplacian in cylindrical coordinates is

$$\nabla^2 = \left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \right],$$

and the differential equation $J_m(k\rho)$ satisfies is

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} - \frac{m^2}{\rho^2} + k^2 \right] J_m(k\rho) = 0.$$

5. (30 minutes) One can demonstrate the reality of the linear momentum of an electromagnetic field by the derivation of the electromagnetic stress tensor

$$T_{ij} = \epsilon_0 (E_i E_j - \frac{1}{2} E^2 \delta_{ij}) + \mu_0 (H_i H_j - \frac{1}{2} H^2 \delta_{ij}),$$

where E_i , H_i are the components of electric field and magnetic intensity, ϵ_0 and μ_0 are the permittivity and permeability of vacuum (SI units), and δ_{ij} is the Kronecker delta. The derivation can be made from ordinary Newtonian mechanics as follows:

$$\vec{F} = \frac{d\vec{p}}{dt} = \iiint_V \rho(\vec{E} + \vec{v} \times \vec{B}) dV = \iiint_V (\rho \vec{E} + \vec{J} \times \vec{B}) dV, \quad (1)$$

where \vec{E} and \vec{B} are the usual electric and magnetic field, $\vec{J} = \rho \vec{v}$ = current density, and $\rho = dQ/dV$ = charge density.

You are given Maxwell's four equations (SI units)

$$\begin{aligned} \nabla \cdot \vec{E} &= \rho / \epsilon_0 & \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} & \nabla \times \vec{B} &= \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \end{aligned}$$

along with $\vec{D} = \epsilon_0 \vec{E}$ and $\vec{B} = \mu_0 \vec{H}$.

(a) Show that Equation (1) above may be written as:

$$\frac{d\vec{p}}{dt} = -\frac{\partial}{\partial t} \iiint_V \vec{D} \times \vec{B} dV + \iiint_V [\vec{E}(\nabla \cdot \vec{D}) - \vec{D} \times (\nabla \times \vec{E}) + \vec{H}(\nabla \cdot \vec{B}) - \vec{B} \times (\nabla \times \vec{H})] dV, \quad (2)$$

(b) Show that the x-component of the electric field contribution from the second integral in (2) is:

$$\begin{aligned} & [\vec{E}(\nabla \cdot \vec{D}) - \vec{D}x(\nabla x \vec{E})]_x = \\ & = \epsilon_0 \left\{ E_x \frac{\partial E_x}{\partial x} + E_x \frac{\partial E_y}{\partial y} + E_x \frac{\partial E_z}{\partial z} - E_y \frac{\partial E_y}{\partial x} + E_y \frac{\partial E_x}{\partial y} + E_z \frac{\partial E_x}{\partial z} - E_z \frac{\partial E_z}{\partial x} \right\}. \end{aligned}$$

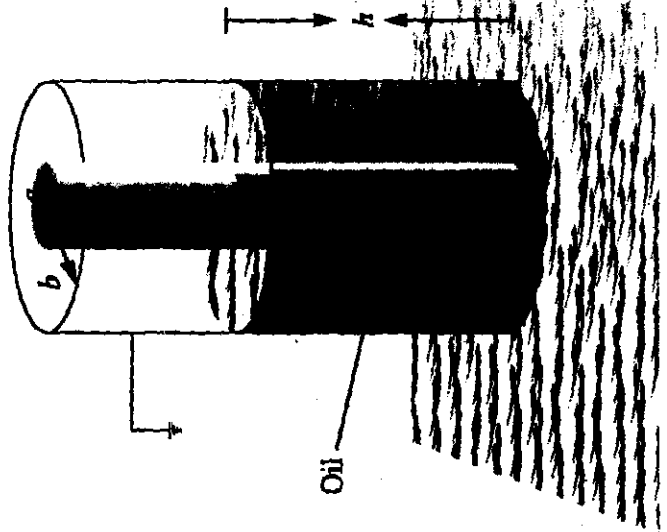
(c) Integrate the expression in (b) over $dV = dx dy dz$ to get the electric field contribution to the electromagnetic stress tensor:

$$\frac{1}{2} \epsilon_0 \iint_{S_x} (E_x^2 - E_y^2 - E_z^2) dS_x + \epsilon_0 \iint_{S_y} E_x E_y dS_y + \epsilon_0 \iint_{S_z} E_x E_z dS_z, \quad (3)$$

where $dS_x = dy dz$, $dS_y = dz dx$, $dS_z = dx dy$.

(Thus, from (2) and (3) $\frac{dp_i}{dt} = -\frac{\partial}{\partial t} \iiint_V [\vec{D}x\vec{B}]_i dV + \sum_{j=1}^3 \iint_{S_j} T_{ij} dS_j$, with $i = 1, 2, 3$, or x, y, z).

6. (30 minutes) Two long coaxial cylindrical metal tubes (inner radius a , outer radius b) stand vertically in a tank of dielectric oil (susceptibility χ_e , mass density ρ). The inner tube is maintained at potential V , and the outer one is grounded (see figure). To what height (h) does the oil rise in the space between the tubes?



2001 Ph.D. Preliminary Examination

Part IV: General, Mathematical & Statistical Physics

Time: 3 hours

Date: May 25, 2001

Instructions: Work five (5) of the following six problems. Work each problem on a separate sheet of paper and turn in only the five problems you wish to have graded. Each problem is designed to require approximately 30 minutes to solve. Use your time wisely and be careful to not spend too much time on any one problem. **Good Luck!**

Note: The student will need a hand calculator for some of the problems below.

1. (30 minutes) In most paramagnetic materials, the individual magnetic particles have more than two independent states (orientations). The number of independent states depends on the particle's angular momentum quantum number j , which must be a multiple of $1/2$. The allowed values of the z component of a particle's magnetic moment are

$$\mu_z = -j \delta_\mu, (-j + 1) \delta_\mu, \dots, (j - 1) \delta_\mu, j \delta_\mu$$

where δ_μ is a constant, equal to the difference in μ_z between one state and the next. Thus the number of magnetic states is $2j + 1$. In the presence of the magnetic field B pointing in the z direction, the particle's magnetic energy (neglecting interactions between dipoles) is $-\mu_z B$.

(a) Show that the partition function of a single magnetic particle is

$$Z = \frac{\sinh[b(j + \frac{1}{2})]}{\sinh \frac{b}{2}}$$

where $b = \beta \delta_\mu B$ and $\beta = (kT)^{-1}$.

(b) Show that the total magnetization of a system of N such particles is

$$M = N \delta_\mu [(j + \frac{1}{2}) \coth(b(j + \frac{1}{2})) - \frac{1}{2} \coth \frac{b}{2}] .$$

(c) Show that the magnetization is proportional to $1/T$ (Curie's law) in the limit

$$T \rightarrow \infty. \quad [\text{Hint : } \coth x \approx \frac{1}{x} + \frac{x}{3} \quad \text{when } x \ll 1.]$$

- 2. (30 minutes)** The Earth's atmosphere is nearly adiabatic because air conducts heat poorly. Combine the pressure balance equation, $dP = -\rho g dh$ (where ρ = density, g = acc. due to gravity, h = height), with the adiabatic law to find an expression for the rate of change of temperature with height, $\frac{dT}{dh}$. Treat air as an ideal gas with $C_v = \frac{3}{2} Nk$ and use $\rho = \frac{N}{V} \mu$, where μ is molecular density.

- 3. (30 minutes)** Solve the following Parabolic partial differential equation with the specified boundary conditions.

$$\frac{\partial U(x,t)}{\partial t} = \alpha^2 \frac{\partial^2 U(x,t)}{\partial x^2}$$

$$\partial U / \partial x = 0 \quad \text{at } x = 0 \text{ for all } t$$

$$\partial U / \partial x = 0 \quad \text{at } x = d \text{ for all } t$$

$$U = f(x) \quad \text{for } t = 0.$$

- 4. (30 minutes)** Consider a system of massive bosons in a harmonic potential trap. (Such systems are now available in the lab and being used to study Bose-Einstein condensation.) Assume that the bosons are noninteracting and that they are in a trapping potential of the form

$$V(\mathbf{r}) = \frac{m}{2} \sum_{i=1}^3 \omega_i^2 r_i^2$$

where m is the mass of the particles, the center of the trap is at the origin and $\omega_i/2\pi$ has a typical value of 60 Hz. (a) What are the single-particle energy levels for this potential? (b) What is the thermodynamic potential? (c) In the isotropic limit ($\omega_0 = \omega_1 = \omega_2 = \omega_3$) show that an expression for the average number of particles is:

$$N = \frac{1}{2} \sum_{m=0}^{\infty} \frac{(m+1)(m+2)}{e^{(m+\varepsilon)t} - 1}$$

where $\varepsilon = 3/2 - \mu/\hbar\omega_0$ and $t = k_B T/\hbar\omega_0$. (d) What is the number of particles in the ground state? Show that your result predicts that the occupation of the ground state will be a function of the temperature.

5. (30 minutes) Consider the integral: $\int_0^{\infty} \frac{x^a}{x+1} dx$ for $-1 < a < 0$.

- The integral is to be evaluated in the complex plane. Identify the singular points and branch points. Specify the degree of any poles.
- Evaluate the integral using complex variables. Describe an appropriate contour. Evaluate the integral along each section of the contour. (You may NOT assume that the integral is zero along any section of the contour.)

6. (30 minutes) a.) For a given square matrix A, compute the matrix

$$B = \frac{1}{(1 - \lambda A)}$$

using Taylor's expansion of $1/(1-x)$.

b.) Show that

$$B(1 - \lambda A) = 1$$

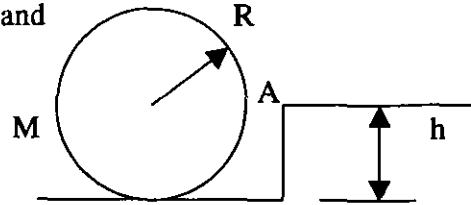
c.) For the case when A can be diagonalized, discuss convergence properties of this expansion. That is, what values of λ will allow the expansion to converge.

2002 Physics Ph.D Preliminary Exam

Section 1: Classical Mechanics

Work five of the following six problems. Work each problem on separate pages.

1. A homogeneous circular disk of mass M , radius R , and angular velocity ω rolls across a horizontal floor and collides with a fixed, raised step of height h ($h < R$). The collision is strictly inelastic and of short duration. Assume that the disk doesn't slip on the edge.



(a) Show that the minimum angular velocity ω_o for the disk to "roll up on the step" is

$$\omega_o = \frac{2\sqrt{gh/3}}{R(1 - 2h/3R)}.$$

(b) What would ω_o be if the rolling object were a homogeneous sphere of mass M and radius R instead of a uniform disk? (Use $I_{\text{sphere}} = \frac{2}{5}MR^2$.)

(c) If $h/R = 0.5$, would the sphere or disk have the higher minimum value of ω_o ? Is this what you would expect? Why?

2. A rigid body has the following tensor of inertia:

$$\mathbf{I} = \begin{pmatrix} 1 & -2 & 5 \\ -2 & 4 & 3 \\ 5 & 3 & 8 \end{pmatrix}.$$

(a) Find the tensor of inertia relative to a parallel set of axes with origin at $\vec{r} = 1\hat{e}_x + 2\hat{e}_y - 2\hat{e}_z$, given that the center of mass is located at the point $\vec{R} = 3\hat{e}_x + 3\hat{e}_y + 2\hat{e}_z$ with respect to the first system. Let $M =$ mass of the body $= 2$ units.

(b) If the angular velocity of the new set of axes is $\vec{\omega}' = 1\hat{e}'_x + 2\hat{e}'_y + 3\hat{e}'_z$ in the appropriate set of units, find the angular momentum \vec{L}' and the kinetic energy T' relative to the new set of axes.

3. A fixed volume V of water is rotating in a cylinder with constant angular velocity ω . Find the curved water surface that minimizes the total potential energy of the water in the combined gravitational-centrifugal force field.

4. A particle is projected vertically upward to a height h above a point on earth's surface at a northern latitude λ . Show that the Coriolis deflection (ΔR) when it again reaches the ground is given by:

$$\Delta R = \frac{4}{3} \omega \cos \lambda \sqrt{\frac{8h^3}{g}}$$

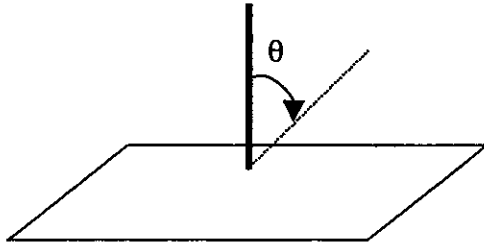
where ω is the rotational frequency of the earth. Neglect air resistance and consider only small vertical heights.

5. A uniform rod of mass m and length l initially stands vertically on a rough horizontal floor and is allowed to fall (i.e. tip-over).

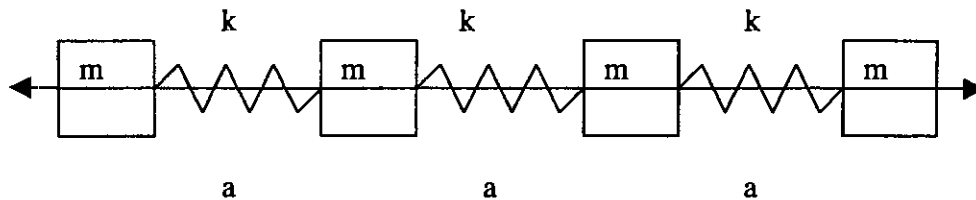
(a) Assuming that slipping has not occurred, determine the rod's angular velocity ω as a function of θ , the angle the rod makes with the vertical.

(b) Also find the normal force exerted by the floor on the rod when the rod has tipped to angle θ from the vertical.

(c) Also find the coefficient of static friction involved if slipping of the rod starts when $\theta = 30$ degrees.



6. Determine the normal modes and frequencies for the following system in 1D.



2002 Physics Preliminary Exam

Section 2: Quantum Mechanics

Work five of the following six problems. Work each problem on separate pages.

1. Positronium is a bound electron-positron system. Neglecting the center-of-mass motion, its Hamiltonian can be written as:

$$H = H_0 + H_1, \text{ where } H_0 = p^2/2m - e^2/r \text{ and } H_1 = b \mathbf{S}_1 \cdot \mathbf{S}_2 .$$

It is assumed that $H_0 \gg H_1$. \mathbf{S}_1 and \mathbf{S}_2 are the electron and positron spins, respectively. Treat b as a constant.

- In terms of the electron mass m_e , what is m in the equation for H_0 ?
- State (or derive) the solutions for the unperturbed spin system. Express the $|s, m, s_1, s_2\rangle$ states as linear combinations of $|s_1, m_1\rangle$ and $|s_2, m_2\rangle$ states. (Assume that \mathbf{L} , the orbital angular momentum, is zero. The total spin quantum number is s , and the z -component is m .)
- Using the results of part b, determine the effect of the perturbation H_1 on the energy levels. Is your result exact or is it correct only to lowest order in perturbation theory?

2. Consider the following one-dimensional problem: Two identical particles with spin 0 and mass, m , are harmonically bound to the origin by a force proportional to the distance to the origin (with force constant k). In addition there is a weak force between the particles proportional to the distance between them (with force constant $k' \ll k$). Use perturbation theory to calculate the energy of the two lowest states of the system.

3. Find the probability that the electron in a hydrogen atom will be found at a distance from the nucleus greater than its energy would permit on the classical theory. Assume the atom to be in its ground state.

Recall that $\int x^n e^{ax} dx = (x^n e^{ax}/a) - (n/a) \int x^{(n-1)} e^{ax} dx .$

4. Consider a He like atom with nuclear charge z and two electrons. (a) Construct all of the spin states that are eigenfunctions of the spin operators S^2 and S_z . Be sure to specify the eigenvalues for each eigenfunction. (b) Write out the Hamiltonian for this two electron system. Construct two wavefunctions which satisfy the exclusion principle for this system, one function for the case when the two electrons have the same spin, and one when they have different spins. Are these wavefunctions eigenfunctions of the spin operators? If not, explain under what conditions they will be eigenfunctions of spin and/or construct wavefunctions that are eigenfunctions. (c) Using the wavefunction for the He atom where the two electrons have the same spin, derive the Hartree-Fock equations for the two electron orbitals. Explain the meaning of any special terms in these equations.

5. Find the bound states for $E < 0$ and a potential

$$V(x) = -V_0 \delta(x-a) - V_0 \delta(x+a), \quad V_0 > 0$$

6. For the state space spanned by the angular momentum sub-levels $l = 1, m_l = 1, 0, -1$

a) determine a matrix representation for L_z, L_+ and L_- , L_x and L_y

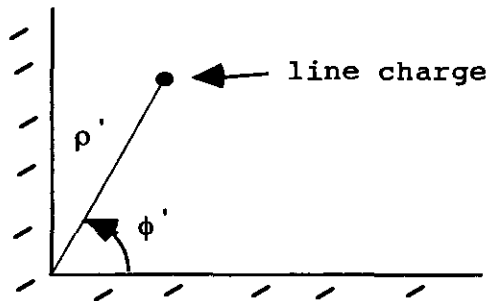
b) show that the representations of L_x, L_y , and L_z obey the angular momentum commutation relations

2002 Physics Prelim

Section 3: E&M

Work five of the following six problems. Work each problem on separate pages.

1. An infinitely long unit line charge is located with cylindrical coordinates (ρ', ϕ') parallel to the line at which perpendicular conducting planes intersect. Using the image method, find the Dirichlet Green function.



Express your answer in *cylindrical* coordinates. Show that

$$G(\vec{x}, \vec{x}') = \ln \left[\frac{\rho^4 + \rho'^4 - 2\rho'^2\rho^2 \cos(2(\phi+\phi'))}{\rho^4 + \rho'^4 - 2\rho'^2\rho^2 \cos(2(\phi-\phi'))} \right].$$

2. In Coulomb gauge, $\vec{\nabla} \cdot \vec{A} = 0$, Maxwell's equations can be written,

$$\begin{aligned} \nabla^2 \Phi &= -\frac{1}{\epsilon_0} \rho, \\ \nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} &= -\mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{\nabla} \Phi}{\partial t}. \end{aligned}$$

Define the transverse current as,

$$\vec{J}_T = \vec{J} - \epsilon_0 \frac{\partial \vec{\nabla} \Phi}{\partial t}.$$

a) Show directly that the transverse current has $\vec{\nabla} \cdot \vec{J}_T = 0$.

b) Show that this current may also be written as

$$\vec{J}_T = \vec{\nabla} \times \left(\vec{\nabla} \times \int \frac{d^3 x'}{4\pi} \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} \right).$$

3. It is only necessary to keep the fields \vec{E} , $\vec{B} \sim o\left(\frac{1}{|\vec{x}|}\right)$ in computing radiation from electromagnetic sources; these are called the acceleration fields, \vec{E}_a , \vec{B}_a . Consider the nonrelativistic motion of a point particle of charge e along a given path, $\vec{w}(t)$. Using the potentials,

$$\Phi(\vec{x}, t) = \frac{e}{4\pi\epsilon_0} \frac{1}{R(1 - \hat{R} \cdot \vec{\beta})}, \quad \vec{A}(\vec{x}, t) = \frac{\vec{\beta}}{c} \Phi,$$

($\vec{\beta} = \frac{\vec{v}}{c}$, $\vec{R} = \vec{x} - \vec{w}(t_r)$) where the quantities on the right are evaluated at the retarded time, t_r ($t_r = t - \frac{|\vec{x}|}{c}$ to the accuracy needed in this problem), and given

$$\vec{E} = -\vec{\nabla} \Phi - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla} \times \vec{A},$$

show that the acceleration fields in nonrelativistic approximation are given by ($\hat{x} = \frac{\vec{x}}{|\vec{x}|}$)

$$\vec{E}_a = \frac{e}{4\pi\epsilon_0 |\vec{x}|^3 c} \hat{x} \times [\hat{x} \times \dot{\vec{\beta}}], \quad \vec{B}_a = \frac{e}{4\pi\epsilon_0 |\vec{x}|^3 c} [\hat{x} \times \dot{\vec{\beta}}].$$

4. A spherical shell, of radius R , carrying a uniform surface charge σ , is set spinning about z axis at angular velocity ω .

- (a) Show that the total dipolar moment (\mathbf{m}) of the shell is:

$$\mathbf{m} = \frac{4\pi}{3} \sigma \omega R^4 \hat{z}$$

- (b) Use multipole expansion of vector potential (\mathbf{A}) to show that:

$$\mathbf{A} = \frac{\mu_0 \sigma \omega R^4}{3} \frac{\sin \theta}{r^2} \hat{\phi}$$

- (c) Find the total angular momentum contained in the electromagnetic field of the spinning shell.

5. A uniform line charge λ is placed on an infinite straight wire, a distance d above a grounded conducting plane. The wire runs parallel to the x -axis, directly above it, and the conducting plane is the xy plane.

- (a) Find the electric potential at a point on yz plane.
 (b) Find the induced surface charge density on the conducting plane.
 (c) What is the total charge induced on a strip of width w parallel to the y -axis?

6. Consider two infinite concentric conducting cylinders of radii R_1 and R_2 ($R_1 < R_2$). Solve the differential equation $\nabla^2 \phi(\vec{r}) = 0$ under the Dirichlet's boundary condition in the domain $R_1 < r < R_2$. Compute capacitances and coefficients of inductors (that is, C_{ij}) for the system, defined by

$$Q_j = \sum_{k=1}^2 C_{jk} V_k \quad (j = 1, 2)$$

Here Q_j is the total electric surface charge of the j -th cylinder (per unit length along the z -axis).

2002 Physics Ph.D Preliminary Exam

Section 4: General Physics

Work five of the following six problems. Work each problem on separate pages

1. The Chebyshev differential equation has the form

$$(1 - x^2) y'' - xy' + n^2 y = 0 .$$

- Put the equation in Sturm-Liouville form. What orthogonality relation will the solutions satisfy?
- Classify all singular points of the equation.
- Find two series solutions of the Chebyshev DE near $x = 0$. Is it necessary to impose restrictions on n if the series is to converge in the entire range $-1 \leq x \leq +1$? If so, what are the restrictions?
- One solution for $n = 1$ is $y_1(x) = x$. Using $y_1(x)$, derive a second, linearly independent solution $y_2(x)$.

2. A square plate has the edges at $x = 0$ and at $x = \pi$ insulated. The edge $y = 0$ is kept at a temperature of zero for $t > 0$, and the edge $y = \pi$ is kept at a temperature of x for $t > 0$. Solve formally for the steady state temperature $U(x,y)$. (Hint: This may be one of those problem where a substitution of the form $U(x,y) = V(x,y) + W(y)$ will help.) Be sure to clearly specify the boundary conditions.

3. A string is tied at the points $x = 0$ and $x = \pi$ with an initial velocity of zero. The initial displacement, y , is given by

$$\begin{aligned} y(x,0) &= h x && \text{for } 0 \leq x \leq \pi/2 \\ &= h (\pi - x) && \text{for } \pi/2 < x \leq \pi \end{aligned}$$

Solve formally for the displacement of the string $y(x,t)$.

4 If $P(B|A) = P(B|A^c)$, show that the two events A and B are statistically independent, where A^c is the complement of the event A .

5. Show that for a PVT system (with fixed N),

$$(\partial S / \partial V)_U (\partial V / \partial T)_S = -S/T \text{ and verify explicitly by considering the ideal gas.}$$

6) (Need Hand Calculator):

(a) Begin with the simple postulates of the Bohr model of the hydrogen atom (an electron e moving in a circular orbit about a proton p) and show that the total energy E_n radius r_n are given by

$$E_n = -\frac{1}{2} m_e \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \frac{1}{\hbar^2 n^2} = -13.6 \text{ eV} / n^2 \text{ and } r_n = \frac{4\pi\epsilon_0 \hbar^2}{m_e e^2} n^2,$$

where (SI units) $e = 1.60 \times 10^{-19} \text{ C}$, $m_e = 9.11 \times 10^{-31} \text{ kg}$, $\hbar = 6.626 \times 10^{-34} \text{ J s}$,
 $\epsilon_0 = 8.85 \times 10^{-12} \text{ C}^2/\text{N m}^2$, $1 \text{ eV} = 1.60 \times 10^{-19} \text{ J}$.

(b) Replace the proton ($m_p = 1836 m_e$) by a positron (same mass as electron, but positive charge) to form "positronium." Find the binding energy in eV for the ground state of positronium, the transition wavelength λ in nm ($1 \text{ nm} = 10^{-9} \text{ m}$) for the Lyman-alpha line ($2p \rightarrow 1s$), and the radius $r_{\text{positronium}}$ of the positronium atom.

(c) The decay rate Γ (energy) for electric-dipole transitions between two states in hydrogen is $\omega^3 |\langle r \rangle|^2$, where $\hbar\omega$ is the energy difference between the two hydrogen states, and r is the relative coordinate between the proton and the electron ($5.29 \times 10^{-11} \text{ m}$). Since $\Gamma\tau \approx \hbar$, where τ is the lifetime for the decay ($\tau \approx 1/\Gamma$), use the results from (a) and (b) to find the lifetime for the $2p \rightarrow 1s$ state in positronium if $\tau = 1.6 \text{ ns}$ for the same transition in hydrogen. (The fact that $\tau = 1.6 \text{ ns}$ for this transition in hydrogen means that an electron in the $2p$ state will only remain there for 1.6 ns).

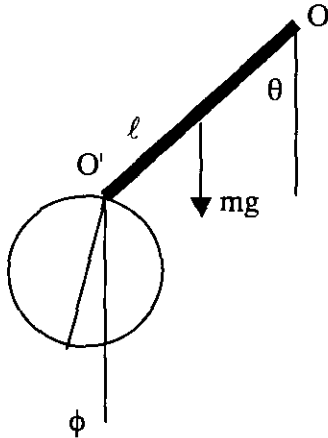
(d) Estimate the value of the magnetic field B in Teslas experienced by the electron due to the positron's magnetic moment μ_{positron} for the $n = 1$ state if

$$\Delta E = \frac{\mu_0}{4\pi} \left(\frac{8\pi}{3} \right) \left(\frac{1}{\pi (r_{\text{positron}})^3} \right) g_s^2 \mu_B^2 = |\mu_B B|,$$

where $\mu_B = \frac{e\hbar}{2m_e}$ is the Bohr magneton (the magnetic moment of an electron in a circular orbit in the hydrogen atom), $g_s = 2$, $r_H = 5.29 \times 10^{-11} \text{ m}$, and $\mu_0 = 4\pi \times 10^{-7} \text{ T m/A}$. Are you surprised at how large B is? Why or why not?

Classical Mechanics Section Summer 2003 Select and solve 5 problems from the 6 given.

1.



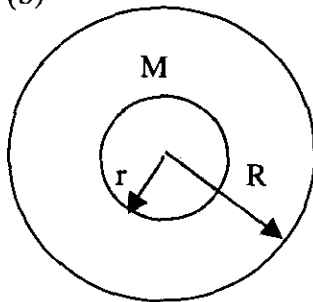
A plane pendulum consists of a uniform rod of length ℓ and mass m , suspended in a vertical plane with one end at O . At the other end of the rod, at point O' , is a uniform disk of mass M and radius a , attached so that it can pivot (rotate) freely in the same vertical plane. Call the angle ϕ that angle between the vertical line through O' and the line M from O' through the center of the disk. Use the angles θ and ϕ to set up the Lagrangian, and from the Lagrangian, determine the two second-order differential equations of motion.

(Use $I_{disk} = \frac{1}{2}Ma^2$; I_{rod} (about center) $= \frac{1}{12}m\ell^2$. Be very careful the way you set up linear and rotational kinetic energies).

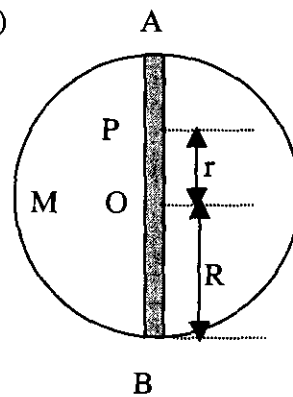
2. (a) If the force between two point masses m_1 and m_2 is $F = Gm_1m_2/r^2$, find the force on a point mass m located at a distance r from the center O of a homogeneous sphere M of radius R , when $r < R$.

(b) Find the potential energy $W(r)$ between \underline{m} and \underline{M} at a distance r ($r < R$) if $W(R) = -GmM/R$ at the surface of the sphere.

(a), (b)



(c), (d)



(c) If a narrow straight tunnel AB is drilled through the center O of the earth as shown in the figure above, show that the point mass \underline{m} will undergo simple harmonic motion. Give the mathematical expression for the period T of the motion.

(d) If $G = 6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2$, $M_{\text{earth}} = 5.98 \times 10^{24} \text{ kg}$, and $R_{\text{earth}} = 6.38 \times 10^6 \text{ m}$, how long in minutes will it take the point mass m to go from B to A ?

3. A particle of mass m is moving in the x - y plane, subject to an inverse-square attractive force, such that $V(r) = -k/r = -k/(x^2 + y^2)^{1/2}$. Start by defining the quantities needed in terms of the x and y rectangular coordinates and then transform to plane polar coordinates, r and θ .

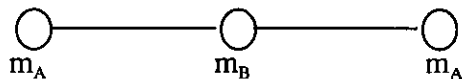
- Using the Lagrangian method, first find the Lagrangian, and from this find the equations of motion for the r and θ variables.
- Now find the Hamiltonian, and using Hamilton's equations, find the resulting equations of motion in terms of the appropriate variables.
- With any time remaining for this problem, solve the resulting equations of motion for either (a) or (b).

4. A pendulum consist of a mass m , in a gravitational field, suspended from a fixed point by a massless string of length b , that is in a viscous medium. The viscous medium produces a retarding force proportional to the velocity, v , given by:

$$F(v) = 2m(g/b)^{1/2} v$$

where g is the acceleration due to gravity. The mass is released from rest at a small initial angular displacement α from the vertical, and allowed to swing back and forth. Find the differential equation that describes the motion of the angular displacement θ , for this damped oscillator. Solve this equation to find the angular displacement, θ , and the angular velocity, $d\theta/dt$. (Note the boundary conditions are: at $t = 0$, $\theta = \alpha$ and $d\theta/dt = 0$.)

5. Consider vibrations of a symmetrical linear triatomic molecule ABA in 1-D.



Consider only modes for which the center of mass is fixed.

That is,

$$m_A(x_1 + x_3) + m_B x_2 = 0.$$

The Lagrangian is

$$L = \frac{1}{2} m_A (\dot{x}_1^2 + \dot{x}_3^2) + \frac{1}{2} m_B \dot{x}_2^2 - \frac{1}{2} k_1 [(x_1 - x_2)^2 + (x_3 - x_2)^2]$$

a) Write the Lagrangian in terms of the new co-ordinates

$$Q_a = x_1 + x_3 \quad \text{and} \quad Q_s = x_1 - x_3$$

b) Determine the two vibrational frequencies.

6. Given the Lagrangian

$$L = -mc^2 \sqrt{1 - \beta^2} - \max$$

where m is the rest mass, $\beta = \frac{\dot{x}}{c}$ and a is the constant magnitude of the force per unit mass.

Set-up and solve the equations of motion for a relativistic particle under the influence of a constant force.

Quantum Mechanics Summer 2003 Work five of the six problems.

1. Imagine a system in which there are only *two* linearly independent states:

$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The most general state is a normalized linear combination:

$|\psi\rangle = a|1\rangle + b|2\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$. Suppose the Hamiltonian is given by $H = \begin{pmatrix} h & g \\ g & h \end{pmatrix}$, where g and h are real constants.

- Find the eigenvalues and (normalized) eigenvectors of the Hamiltonian.
- Suppose the system starts out (at $t = 0$) in state $|1\rangle$. Using the time-dependent Schrödinger equation determine the state of the system at a later time t .

2. The $n = 3$ level of hydrogen is degenerate. Include electron spin and answer the following questions, providing diagrams of the energy splittings for b) through e).

- What is the total degeneracy for the $n = 3$ level?
- What part of the degeneracy is removed by the perturbation $W = A L_z$, with coefficient A being a constant? What is the degeneracy of each sublevel?
- Same question as under b) for $W = A L$.
- Same question for $W = A r^2$, knowing that $\langle r^2 \rangle = (1/2) [5 n^2 + 1 - 3 L (L + 1)] n^2 a^2$.
- Same question for $W = A z$ (the Stark effect).

3. A 1-d harmonic oscillator at $t = 0$ is in a state which is a superposition

$$\psi(x,0) = (1/2) u_0 + (1/\sqrt{2}) u_1 + (1/2) u_2,$$

with u_0 , u_1 , and u_2 being the normalized harmonic oscillator ground state and first and second excited states, respectively. Find $\langle x^2 \rangle$ for all later times. Find the time average of $\langle x^2 \rangle$.

4. The 4 states of two spin 1/2 angular momenta can be represented as $|++\rangle$, $|+-\rangle$, $| -+\rangle$, and $|--\rangle$, where the + or - indicates a spin up or spin down angular state (along z), respectively.

(a) Construct the states of squared total angular momentum, $S^2 = (S_1 + S_2)^2$, and total z-component, $S_z = S_{1z} + S_{2z}$, from these 4 states. Demonstrate that 3 of them have eigenvalue $S^2 = 2\hbar^2$ and one has $S^2 = 0$. [Note that $S_+|s,m\rangle = \sqrt{(s-m)(s+m+1)}|s,m+1\rangle$ and $S_-|s,m\rangle = \sqrt{(s+m)(s-m+1)}|s,m-1\rangle$, where $S_{\pm} = S_x \pm iS_y$.]

(b) The states in (a) are rotated by an angle θ about the z-axis. Find the new states after the rotation. Do these still have the same S^2 and S_z eigenvalues as before?

5. Using the semi-classical treatment of radiation, the rate of spontaneous emission can be obtained as $R_{if} = 4/3(\omega_{if}^3/c^2)\alpha|r_{if}|^2$, where α is the fine structure constant, $e^2/\hbar c$.

Calculate the rate of spontaneous emission of radiation in the dipole approximation from a one-dimensional harmonic oscillator carrying a charge e .

6. For a Dirac particle with positive energy, construct properly normalized wave functions, which are eigenstates of the helicity operator h and correspond to the two eigenvalues ± 1 of h . In Dirac representation,

$$h = \frac{1}{|\vec{p}|} \begin{pmatrix} \vec{\sigma} \cdot \vec{p} & 0 \\ 0 & \vec{\sigma} \cdot \vec{p} \end{pmatrix}$$

1. Infinitely long cylindrical shell of radius R carries a circumferential uniform surface current $\vec{K} = k\hat{\phi}$. Use the Biot-Savart law in the form of

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{K(\vec{r}') \times \hat{r}}{r^2} da'$$

and show that the magnetic field at a distance d from the cylinder axis is given by

$$\vec{B} = \frac{\mu_0}{2} k \left[\frac{R^2 - d^2}{|R^2 - d^2|} + 1 \right] \hat{z} .$$

Use this result to find the magnetic field inside and outside an infinitely long solenoid of radius R , with n turns per unit length, carrying a steady current I .

2. A beam of light is incident normally from air (refractive index n_1) on a plane slab of a transparent dielectric with refractive index n_2 and thickness h . The light passes through the slab and enters a third medium with refractive index n_3 and infinite extent. Show that the reflecting coefficient of the slab is given by

$$R = \frac{r_{12}^2 + r_{23}^2 + 2r_{12}r_{23} \cos 2k_2 h}{1 + r_{12}^2 r_{23}^2 + 2r_{12}r_{23} \cos 2k_2 h}$$

where $r_{12} = \frac{n_2 - n_1}{n_2 + n_1}$, $r_{23} = \frac{n_3 - n_2}{n_3 + n_2}$ and k_2 is the wave number for the

dielectric medium. Assume all the media are nonconducting and nonmagnetic ($\mu = 1$). Use R to find the condition for zero reflection back into the first medium. This is the principle of coated optical lenses.

3. A sphere of radius R has a uniform charge density ρ except for an empty bubble of radius $r < R$ at a distance d from the center. The sphere is located a distance D from a grounded plane sheet as shown. Find the attractive force between sphere and plane.

4. (a) Show that the normal force, $f \cdot n$, (n directed outward from the material) on a vacuum/dielectric interface is given by (cgs units)

$$f \cdot n = 2\pi \sigma_b^2 \left(\frac{\epsilon + 1}{\epsilon - 1} \right) \left(SI: \frac{1}{2\epsilon_0} \sigma_b^2 \left(\frac{\epsilon + \epsilon_0}{\epsilon - \epsilon_0} \right) \right),$$

where σ_b is the bound surface charge and ϵ is the dielectric constant.

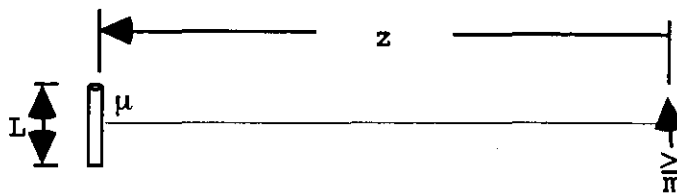
- (b) Similarly, show that the normal force, $f \cdot n$, on a vacuum/magnetic interface is given by

$$f \cdot n = \frac{2\pi}{c^2} K_b^2 \left(\frac{\mu + 1}{\mu - 1} \right) \left(SI: \frac{1}{2\mu_0} K_b^2 \left(\frac{\mu + \mu_0}{\mu - \mu_0} \right) \right),$$

where K_b is the bound surface current and μ is the $\Delta W = \frac{1}{2} \int d^3x M \cdot B_0$,

where M is the magnetization and B_0 is the field before the introduction of the cylinder. Find the force of the dipole on the cylinder. Are they attracted or repelled for $\mu > 1$ (SI: $\mu/\mu_0 > 1$)?

5. The middle of a long, thin cylinder of radius a and length L , with permeability μ is located a large distance z from a point magnetic dipole, m ($z \gg L \gg a$). It is oriented with its long lengthwise dimension oriented perpendicular to the direction to the dipole, but along m , as shown.



The interaction field energy of the configuration is (cgs or SI)

6. Determine the monopole, dipole and quadrupole moments about the origin for a point charge, q , a distance, a , from the origin along the $+z$ -axis.

Use the equation

$$q_{lm} = \int Y_{lm}(\theta', \phi') r'^{\ell} \rho(\vec{x}') d^3 x'$$

and

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta * e^{i\phi}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{22} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2\theta * e^{2i\phi}$$

$$Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin\theta * \cos\theta * e^{2i\phi}$$

$$Y_{20} = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2\theta - \frac{1}{2}\right)$$

$$Y_{l(-m)} = (-1)^m Y_{lm}^*$$

Mathematical Methods/ Statistical Mechanic Summer 2003 Work five of the six problems.

1. An unknown function satisfies the differential equation

$$y''(x) + k^2 y(x) = 0$$

subject to the boundary conditions: $y(0) = 1$, and $y(1) = 0$.

- Determine the normalized eigenfunction corresponding to the smallest eigenvalue.
 - Using the trial function $\phi(x) = 1 - x^n$, determine the optimum value of n and the approximate eigenvalue λ . How does your approximate eigenvalue compare with λ_{exact} ?
2. Consider the one-dimensional, homogeneous heat-flow equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

for a rod of length L . The ends are maintained at zero temperature and the initial temperature everywhere else is $u_0 = \text{constant}$. Find $u(x,t)$.

3. In most paramagnetic materials, the individual magnetic particles have more than two independent states (orientations). The number of independent states depends on the particle's angular momentum quantum number j , which must be a multiple of $1/2$. The allowed values of the z component of a particle's magnetic moment are

$$\mu_z = -j \delta_\mu, \quad (-j + 1) \delta_\mu, \quad \dots, \quad (j - 1) \delta_\mu, \quad j \delta_\mu$$

where δ_μ is a constant, equal to the difference in μ_z between one state and the next. Thus the number of magnetic states is $2j + 1$. In the presence of the magnetic field B pointing in the z direction, the particle's magnetic energy (neglecting interactions between dipoles) is $-\mu_z B$.

- (a) Show that the partition function of a single magnetic particle is

$$Z = \frac{\sinh[b(j + \frac{1}{2})]}{\sinh \frac{b}{2}}$$

where $b = \beta \delta_\mu B$ and $\beta = (kT)^{-1}$.

- (b) Show that the total magnetization of a system of N such particles is

$$M = N \delta_\mu [(j + \frac{1}{2}) \coth\{b(j + \frac{1}{2})\} - \frac{1}{2} \coth \frac{b}{2}] .$$

- (c) Show that the magnetization is proportional to $1/T$ (Curie's law) in the limit

$$T \rightarrow \infty. \quad [\text{Hint : } \coth x \approx \frac{1}{x} + \frac{x}{3} \quad \text{when } x \ll 1.]$$

4. An integral equation $\psi(x_2, t_2) = \int_{-\infty}^{+\infty} K(x_2 - x_1; t_2 - t_1) \psi(x_1, t_1) dx_1$ can be used to determine the wave

function $\psi(x_2, t_2)$ in terms of that at x_1, t_1 if the function K is known. You are given that

$$K(\rho, \tau) = e^{-\pi i/4} \left(\frac{m}{2\pi\hbar\tau}\right)^{1/2} e^{i\left(\frac{m\rho^2}{2\hbar\tau}\right)} \text{ and } \psi(x, 0) = (\pi\xi_0^2)^{-1/4} \exp\left(\frac{ip_0x}{\hbar}\right) e^{-\left(\frac{im\rho^2}{2\xi_0^2}\right)}, \text{ where } p_0 \text{ is the}$$

momentum.

Use the given information to show that

$$\psi(x, t) = \frac{1}{(\pi\xi_0^2)^{1/4} \sqrt{1 + \frac{i\hbar t}{m\xi_0^2}}} e^{\frac{imx^2}{2\hbar}} e^{\frac{it}{2m\hbar} \left(\frac{p - mx/t}{1 + \frac{i\hbar t}{m\xi_0^2}} \right)^2}$$

(It can easily be seen--but don't try to show it!--that from the absolute value squared of this wavefunction, that the wave packet spreads with time as $\xi = \xi_0 \sqrt{1 + \frac{\hbar^2 t^2}{m^2 \xi_0^4}}$).

5. Show that the electric polarization P of an ideal gas consisting of N diatomic molecules having a constant electric dipole moment μ is given by

$$P = \frac{N}{V} \mu \left\{ \coth\left(\frac{\mu E}{kT}\right) - \frac{kT}{\mu E} \right\}$$

where V is the volume of the gas and E is the external electric field.

6. A semi-infinite slab, $0 \leq x \leq a$, is subject to the temperature conditions:

(a) $\frac{\partial U}{\partial t} = k \frac{\partial^2 U}{\partial x^2}$

(b) $U(0, t) = 0$

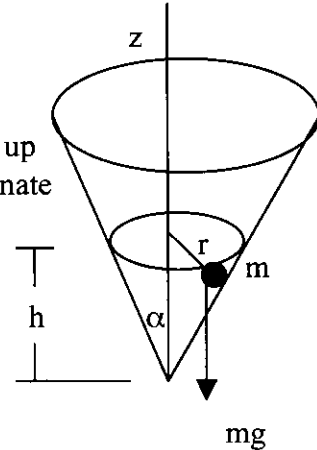
(c) $U(a, t) = 1 + \cos \omega t$

Find the complex function of which the steady periodic component of the temperature is the real part. (Hint: Change the c condition above, to $1 + e^{i\omega t}$.) Show that the complex solution satisfies the equation and the boundary conditions.

Ph.D. Preliminary Examination of 2004
Monday, May 24, 2004

PART I: Classical Mechanics (9:00 a.m. to 12:00 p.m.)

1. (30 minutes) A particle of mass m moves on the inner surface of a smooth right circular cone of half angle α as shown in the figure. Set up the problem in cylindrical coordinates (r, θ, z) . Use $z = r/\tan\alpha$ to eliminate z and \dot{z} .



(a) Obtain the Lagrangian

$$L(r, \theta, \dot{r}, \dot{\theta}) = \frac{m}{2} \left[\frac{\dot{r}^2}{\sin^2 \alpha} + r^2 \dot{\theta}^2 \right] - \frac{mgr}{\tan \alpha}, \text{ and from this show}$$

$$\text{that } \ddot{r} = r \sin^2 \alpha \dot{\theta}^2 = -g \sin \alpha \cos \alpha \quad (1)$$

$$\text{and } \frac{d}{dt}(mr^2\dot{\theta}^2) = \frac{d\ell}{dt}, \text{ or } \ell = mr^2\dot{\theta} = \text{constant} \quad (2)$$

(b) Suppose the mass is projected in a horizontal circle of radius r_0 , and tangential velocity $v_0 = \sqrt{8gh/3}$, where h is the height shown in the figure. Substitute $\dot{\theta} = \ell/(mr^2)$ in Eq. (1)

$$\text{above, and use } \ddot{r} = \dot{r} \frac{d\dot{r}}{dr} \text{ to get } \int r \dot{r} d\dot{r} - \frac{\ell^2 \sin^2 \alpha}{m^2} \int r^{-3} dr = -g \sin \alpha \cos \alpha \int dr + C, \quad (3)$$

where C is an integration constant. Then use $r_0 \dot{\theta}_0 = v_0 = \sqrt{8gh/3}$, $\ell^2 = m^2 h^2 \tan^2 \alpha 8gh/3$ to integrate Eq. (3) and obtain $C = \frac{7gh}{3} \sin^2 \alpha$ from the initial conditions.

(c) Show that the resulting equation can be reduced to

$$\frac{\dot{r}^2}{2hg \sin^2 \alpha} = \frac{7}{3} - \frac{4}{3} \frac{h^2 \tan^2 \alpha}{r^2} - \frac{r}{h \tan \alpha} = \frac{7}{3} - \frac{4}{y^2} - y, \quad (4)$$

where $y = \frac{h}{r \tan \alpha}$. Although it is not obvious, show that the quantity

$$\frac{(1-y)(y-2)(3y+2)}{3y^2}, \text{ when multiplied out, gives the same value } \frac{7}{3} - \frac{4}{y^2} - y. \text{ This means}$$

that the two roots of Eq. (4) are $y = \frac{r}{h \tan \alpha} = 1$ or 2 , or that $r = h \tan \alpha$, or $r = 2h \tan \alpha$. This

means that the particle oscillates periodically between two horizontal circles of radius $h \tan \alpha$ or $2h \tan \alpha$. (Fascinating!)

2. (30 minutes) The tensor of inertia I for a certain object is given by:

$$I = \begin{bmatrix} 7 & \sqrt{6} & -\sqrt{3} \\ \sqrt{6} & 2 & -5\sqrt{2} \\ -\sqrt{3} & -5\sqrt{2} & -3 \end{bmatrix}$$

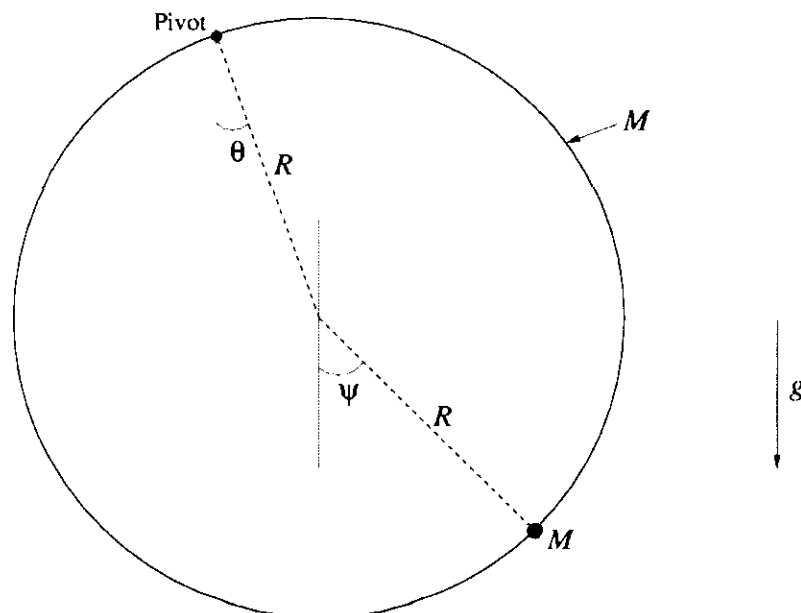
- (a) Diagonalize the tensor and thereby find the principal moments of inertia.
- (b) Determine the value of any two of the principal axes. (You will not have time to find all three). Make sure the eigenvectors are appropriately normalized.

3. A hoop of mass M and radius R is pivoted from one point in a uniform gravitational field as shown in the figure. In addition, a bead of mass M slides without friction along the hoop.

- (a) Using θ and ψ as generalized coordinates, show that the Lagrangian in the small angle approximation $\theta \ll 1$, $\psi \ll 1$ is:

$$L(\theta, \psi, \dot{\theta}, \dot{\psi}) \cong \frac{1}{2}MR^2 \left[3\dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\theta}\dot{\psi} \right] + MgR \left[3 - \theta^2 - \frac{\psi^2}{2} \right]$$

- (b) Find the equations of motion for this system.
- (c) Find the eigenfrequencies of oscillation.
- (d) Sketch the normal modes of oscillation.



4. Consider a particle of rest mass m moving non-relativistically in an attractive potential $V = -\alpha/r$, where α is a positive constant. Here, r is the usual radial coordinate in a spherical coordinate system with origin \vec{O} , so that the total position vector $\vec{x} = (r, \theta, \phi)$ of the particle is with the usual identifications of θ and ϕ as the polar and azimuthal angles, respectively.

(a) Write down the Lagrangian for the system.

(b) Derive the equations of motion for the system.

(c) Show that, when the energy E of the system is negative, the motion of the particle can be represented as

$$p/r = 1 + e \cos \phi,$$

where $p = L^2/(m\alpha)$ and $e = \sqrt{1 + 2EL^2/(m\alpha^2)}$, for some constant L , and give a physical interpretation of L .

5. A particle in a central potential $V = Kr$. In terms of the particle's energy and angular momentum, determine the radius of the circular orbit and the frequency of small oscillations around such an orbit. Eliminate K from your answers. Is the orbit closed according to your approximate calculation?

6. A wheel rolls without slipping along a straight line in a horizontal plane. It has mass M and moment of Inertia I around its center and radius R . A pendulum of mass m suspended by a string of length s ($< R$) from the hub of the wheel. Assuming that this pendulum swings only in the plane of the wheel (without hitting anything), derive a Lagrangian and equations of motion for the system. Find two constants of the motion.

Ph.D. Preliminary Examination of 2004

Monday, May 24, 2004

PART II: Quantum Mechanics (1:00 p.m. to 4:00 p.m.)

1. a) A potential is given by $V(\mathbf{r}) = a x^2 + b y^2 + c z^2$, where a , b , & c are constants. What condition(s) must a , b , & c fulfill, so that L_z , associated with a particle in the potential, will be a constant of motion? (15 points)

b) The wave function of a particular of mass m moving in an unknown potential well is

$$\Psi = (x + y + z) \exp[-a(x^2 + y^2 + z^2)], \text{ where } a \text{ is a positive real constant.}$$

Calculate the probability of obtaining for a measurement of L^2 and L_z , the values $l = 2$ and $m = 0$, respectively. (15 points)

2. a) A nucleus has spin $7/2 \hbar$. How many independent measurements are necessary to determine its spin state? How many independent pure spin states are there? (15 points)

b) Three particles are identical spin- S bosons. Their total spatial wave function is symmetric. Two of these bosons have the same z -component of spin, denoted m_a , and one has a different z -component of spin, denoted m_b . Write the total spin wave function for these three bosons in terms of the individual spin wave functions $|m_a\rangle$, $|m_a\rangle$, and $|m_b\rangle$. (8 points)

c) Repeat b), for the case of the three particles being identical spin- S fermions. (7 points)

3. Consider a particle of mass m and charge q constrained to move on a circle of radius R .

(a) What is the Hamiltonian H_0 in the absence of an electric field?

(b) What are the energy eigenstates and their energies?

(c) If an electric field \mathbf{E} is turned on in the $\theta = 0$ direction, the Hamiltonian is perturbed by an interaction term $H_1 = -q \mathbf{E} R \cos \theta$. What is the ground state wave function to first order in \mathbf{E} ?

(d) What is the energy of the ground state to order \mathbf{E}^2 ?

(e) What is the expectation value of the charge distribution in the perturbed ground state, to order \mathbf{E} ?

(f) The charge distribution will create an induced electric field E_q . What is the ratio $E_q(r=0)/E$ of the induced field at the center of the ring to the external field, when the charge q is in the perturbed ground state?

4. Let $|i\rangle$, ($i = 1, 2, 3$) be eigenstates of a Hamiltonian operator A with different eigenvalues.

The Hamiltonian operator is given by

$$H = \Delta (|1\rangle\langle 2| + |2\rangle\langle 1| + 2|3\rangle\langle 3|),$$

where Δ is a real and positive constant. (a) Find the eigenvalues and the corresponding eigenstates of H . (b) Suppose that the probabilities of finding the system in the states $|1\rangle$, $|2\rangle$, and $|3\rangle$ are $1/2$, 0 , and $1/2$, respectively at the initial moment $t = 0$. Using the Schrodinger equation write down the state vector $|\psi(t)\rangle$.

5. Consider the scattering of a particle with mass m by a spherical potential $V(r)$ given by

$$V(r) = \begin{cases} \infty, & r < a \\ -V_0, & a < r < R \\ 0, & r > R \end{cases}$$

where a , R , and V_0 are all real and positive constants. Write down the equations that determine the phase shifts δ_ℓ ($\ell = 0, 1, 2, \dots$) [You may directly use the expression for the radial wave function A_ℓ for $r > R$

$$A_\ell(r > R) = e^{i\delta_\ell} [\cos(\delta_\ell)j_\ell(kr) - \sin(\delta_\ell)n_\ell(kr)]$$

where $k \equiv (2mE)^{1/2}/\hbar$ and $j_\ell(x)$ and $n_\ell(x)$ are the spherical Bessel functions.].

6. A charged particle linear harmonic oscillator is in a time-dependent homogeneous electric field given by

$$E(t) = \frac{A}{\sqrt{\pi\tau}} \exp\left(-\frac{t^2}{\tau}\right)$$

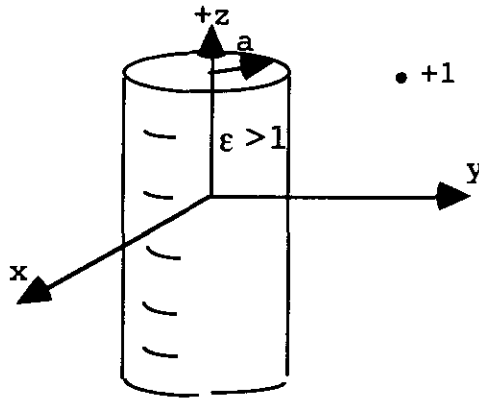
where A & τ are constants. If at $t = -\infty$, the oscillator is in its ground state, find, to a first approximation, the probability that it will be in its first excited state at $t = +\infty$.

Ph.D. Preliminary Examination of 2004

Tuesday, May 25, 2004

PART III: Electricity & Magnetism (9:00 a.m. to 12:00 p.m.)

1. (a) Consider an infinitely long dielectric ($\epsilon > 1$) cylinder of radius a , with the z -axis of coordinates along the axis of symmetry of the cylinder.



Assuming the reduced form for the Green function,

$$G(\vec{x}, \vec{x}') = \frac{2}{\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \int_0^{\infty} dk \cos [k(z-z')] g_m(\rho, \rho'),$$

and with

$$4\pi\delta(\vec{x} - \vec{x}') = \frac{2}{\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \int_0^{\infty} dk \cos [k(z-z')] \frac{1}{\rho} \delta(\rho-\rho'),$$

establish the differential equation satisfied by $g_m(\rho, \rho')$ for the case of the point charge exterior to the cylinder.

(b) Given the general form of solutions in the three regions,

$$g_m(\rho, \rho') = \begin{cases} A_m I_m(k\rho), & \rho < a, \\ B_m I_m(k\rho) + C_m K_m(k\rho), & a < \rho < \rho', \\ D_m K_m(k\rho), & \rho' < \rho. \end{cases}$$

where $K_m(k\rho)$, $I_m(k\rho)$ are (linearly independent) Bessel functions of imaginary argument, apply the continuity and boundary conditions to get the equations which determine the coefficients. (No need to solve these.)

2. (a) Starting with the instantaneous nonrelativistic dipole radiation formula,

$$\frac{dP(t)}{d\Omega} = \frac{e^2}{4\pi c^3} |\hat{n} \times \ddot{\mathbf{x}}(t_0)|^2,$$

($t_0=t-r/c$) and defining

$$\int_0^\infty d\omega \frac{d^2E}{d\omega d\Omega} \equiv \int_{-\infty}^\infty dt \frac{dP}{d\Omega}(t),$$

show that the angular/frequency spectrum is

$$\frac{d^2E}{d\Omega d\omega} = \frac{e^2}{2\pi c^3} |\hat{n} \times \ddot{\mathbf{x}}(\omega)|^2,$$

where

$$\ddot{\mathbf{x}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt e^{i\omega t} \ddot{\mathbf{x}}(t_0).$$

(b) For uniform acceleration, a , along \hat{k} ,

$$\ddot{\mathbf{x}}(t_0) = \begin{cases} 0, & t_0 < 0, \\ a \hat{k}, & 0 < t_0 < T, \\ 0, & T < t_0 \end{cases}$$

and using part (a), find the angular/frequency energy cross section, $\frac{d^2E}{d\Omega d\omega}$, for this process.

3. As a simple example of grain charging in a complex plasma, assume that only primary charging mechanisms (electron & ion capture by the grain) need be considered for a single neutral grain, at rest in a Maxwellian two-component plasma with $c_{si} \ll c_{sc}$. Here $c_{s\alpha}$ stands for the different thermal velocities, in this simple case given through

$$c_{s\alpha}^2 = \kappa T_\alpha / m_\alpha$$

with κ being Boltzmann's constant and T_α referring to the respective plasma species temperatures.

(i) With $f_\alpha(\mathbf{v})$ representing the velocity distribution at infinity, argue that the charging current to the grain due to plasma species α is given by

$$I_\alpha = n_\alpha q_\alpha \int_{v_0}^\infty v \sigma_\alpha f_\alpha(v) d^3v$$

where σ_α is the charging cross section and v_0 is the smallest particle velocity required to hit the grain.

- (ii) Assume that for a $\ll \lambda_D$ (where a is the grain radius and λ_D the Debye shielding length) the capacitance of the grain is given by

$$C = 4\pi\epsilon_0 a$$

so the dust charge can be written as

$$Q_D = 4\pi\epsilon_0 aV$$

where V is the grain potential. Find the charging cross section, σ_α . (HINT: Remember your classical mechanics and use b_c as your impact parameter. Express your answer in terms of a^2 , q_α , m_α , V and v^2 .)

- (iii) When $q_\alpha V < 0$, the particle and grain attract each other. On the other hand, for $q_\alpha V > 0$, plasma particles must overcome the potential barrier of the grain to be included in the charging process. Use your result from (ii) along with energy conservation to find a general expression for I_α for both an attractive potential ($q_\alpha V < 0$) and a repulsive potential ($q_\alpha V > 0$). (HINT: Use spherical coordinates and assume that the velocity distribution (f_α) can be taken as Maxwellian at sufficiently large distances.)

- (iv) For an isothermal, two component plasma show that the grain will be negatively charged and that I_i and I_e are given by

$$I_i = \pi a^2 n e \sqrt{\frac{8\kappa T}{\pi m_i}} \left(1 - \frac{eV}{\kappa T}\right) \quad I_e = -\pi a^2 n e \sqrt{\frac{8\kappa T}{\pi m_e}} \exp\left[\frac{eV}{\kappa T}\right]$$

4. A point charge q of mass m is released from rest at a distance d from an infinite grounded conducting plane.

- (a) What is the force acting on the point charge at the moment of release?
 (b) How long will it take for the point charge to hit the plane?
 (c) What is the speed of the point charge when it hits the plane?

5. Consider a closed region of space V with bounding surface S . Prove the uniqueness of the solution of Poisson's equation,

$$\nabla^2 \Phi(\vec{x}) = -\rho(\vec{x})/\epsilon_0,$$

inside V subject to the boundary condition $\Phi|_S = \Phi_0(\vec{x}(\sigma))$, where $\Phi_0(\vec{x}(\sigma))$ is a given function on S with S parametrized by $S = \{\vec{x}(\sigma) : \sigma = (\sigma_1, \sigma_2), 0 \leq \sigma_i \leq a_i, i = 1, 2\}$ for some constants a_i , $i = 1, 2$. Here, ρ is the charge density in V .

6. a. An infinitely long cylinder, of radius R , carries a "frozen-in" magnetization parallel to the axis,

$$\vec{M} = kr\hat{z},$$

where k is a constant and r is the distance from the axis. Find the magnetic field inside and outside the cylinder assuming that there is no free current anywhere in the cylinder.

b. A uniformly charged solid sphere of radius R carries a total charge Q , and is set spinning with angular velocity ω about the z axis. Find the dipolar moment of the sphere.

c. Show that the magnetic field at a point (r, θ, ϕ) far away from a dipole $\vec{m} = m\hat{z}$ is given by

$$B_{dip}(r) = \frac{\mu_0 m}{4\pi} (2\cos\theta\hat{r} - \sin\theta\hat{\theta}).$$

Ph.D. Preliminary Examination of 2004

Tuesday, May 25, 2004

PART IV: General Physics, Math Physics, Statistical Mechanics (1:00 p.m. to 4:00 p.m.)

1. Consider a rubber band whose length without tension at temperature T is given by $L_0(T)$. If the rubber band is placed under tension at temperature T , it is found that the tension on the band is given by

$$F = \kappa T [(L/L_0) - (L_0/L)^2].$$

This is the equation of state for the rubber band. The specific heat at constant length is $C_L = (dQ/dT)_L$. The thermodynamic relation $dU = TdS + FdL$ and the Maxwell relation $(dS/dL)_T = -(dF/dT)_L$ may be useful.

- (a) Derive an expression for the change in internal energy when the band is stretched at constant temperature: $(dU/dL)_T$. Show that your expression is proportional to the coefficient of linear expansion at zero tension, $\alpha_0 = (1/L_0)(dL_0/dT)$.
- (b) Calculate the coefficient of linear expansion at constant tension, $\alpha_F = (1/L)(dL/dT)_F$ as a function of T , L/L_0 , and α_0 .
- (c) Show that α_F changes sign at a critical value of L/L_0 . Find an expression for this value of L/L_0 as a function of the dimensionless parameter $\alpha_0 T$.
- (d) What is the change in entropy if the rubber band is stretched from length L_0 to length L_1 at a fixed temperature T_0 ?
- (e) What is the change in entropy if the rubber band is heated from temperature T_0 to temperature T_1 while at constant length L_1 ?
- (f) Find the ratio of temperatures T_1/T_0 when the rubber band is adiabatically stretched from length L_0 to L_1 .

2. The momentum distribution in a gas, which has particle density (N/V) , is

$$f(\vec{P}) = (2\pi mkT)^{-3/2} e^{-P^2/2mkT} (1 + \epsilon \cos \alpha)$$

where $\epsilon \ll 1$ and α is the angle between \vec{P} and the x axis.

- (a) Compute the mean drift velocity \vec{U} of the gas.
- (b) How many atoms per second are crossing, in the positive x direction, a unit area

of the y-z plane? How many are crossing in the negative direction? How are these quantities related to the x-component of \vec{U} ?

3. Consider a system of N ($\gg 1$) weakly interacting harmonic oscillators, forming an isolated system with total energy U . The oscillators are one-dimensional and all have the same frequency ω .

(a) What is the temperature T of the system? What is the physical origin of this temperature and what is the heat bath which maintains it?

(b) By exploiting the relationship between U (the internal energy) and T , show that the probability $P(n)$, that a particular oscillator will have the energy $\hbar\omega(n + 1/2)$ at a given instant, is given by

$$P(n) = \left[1 - e^{-\beta\hbar\omega} \right] e^{-n\beta\hbar\omega}$$

Evaluate $\sum P(n)$ and $\sum nP(n)$.

4. A crystal contains ions in a $J = 1$ level. An internal electric field partly removes the $(2J + 1)$ degeneracy of the $J = 1$ level, resulting in a doublet at energy zero and a single state at energy $D > 0$. When a small magnetic field $\mathbf{H} = H \mathbf{k}$ is applied along the z-axis, the energies of these states become

$$0; \quad -(g\mu_0 H)^2/D; \quad D + (g\mu_0 H)^2/D \quad (g\mu_0 H \ll D)$$

(a) Find the partition function for such an ion and the magnetic moment for a system of N such (non-interacting) ions.

(b) Find, to the leading term in the appropriate expansion, the high temperature and low temperature limits of the magnetic susceptibility. Comment on, and explain, the different D -dependence in the two cases.

5. Use the calculus of residues to evaluate the integral:

$$\int_{-\infty}^{\infty} \frac{\cos ax}{(1+x^2)^2} dx$$

Distinguish the three cases: $a > 0$, $a = 0$, and $a < 0$. Justify each step.

6. Find the solution of the following nonhomogeneous second-order differential equation subject to the boundary conditions $y(0) = 0$ and $y'(0) = 1$.

$$y''(x) + y(x) = xe^x$$

2005 Preliminary Examination

Part I: Classical Mechanics

Time: 3 hours

Date: May 26, 2005

Instructions: Work five (5) of the following six problems. Work each problem on a separate sheet of paper and turn in only the five problems you wish to have graded. Each problem is designed to require approximately 30 minutes to solve. Use your time wisely and be careful to not spend too much time on any one problem. Good Luck!

1. Hamiltonian Mechanics: Consider a particle of mass m that is constrained to move on the surface of a cylinder that is defined by the equation:

$$x^2 + y^2 = R^2$$

(i.e. R is the radius of the cylinder and the origin is located at the geometric center of the cylinder.) The particle is subject to a force that is directed toward the origin and is proportional to the distance the particle is from the origin:

$$\mathbf{F} = -k \mathbf{r}$$

This problem is to be solved by using Hamilton's Dynamics and it is recommended that you use cylindrical coordinates.

- Find the potential corresponding to this force, the kinetic energy and from these the Lagrangian.
- Find the Hamiltonian and then find Hamiltonian's equations of motion.
- Find the general form of the solution to the resulting equation of motion.
- Find the specific solution when the particle is released at a height z_0 above the x - y plane with zero vertical velocity, and at $\theta = 0$ and an angular velocity of 2π rad/sec.

2. A thin hoop of radius R and mass M oscillates in its own plane with one point of the hoop fixed. Attached to the hoop is a small mass m constrained to move (in frictionless manner) along the hoop. Consider only small oscillations, and show that the eigenfrequencies are

$$\omega_1 = \sqrt{\frac{2g}{R}} \quad \text{and} \quad \omega_2 = \sqrt{\frac{g}{2R}} .$$

Find the normal modes and describe the physical situation for each mode.

3.(a) Construct the relativistically correct Hamiltonian ($\vec{A} = \vec{A}(\vec{x}, t)$, $\Phi = \Phi(\vec{x}, t)$) for a charged particle

$$H = \sum_k p^k \dot{x}^k - L_R,$$

where

$$p^k = \frac{\partial L_R}{\partial \dot{x}^k},$$

for the Lagrangian

$$L_R = -\frac{1}{\gamma} mc^2 + \frac{q}{c} \dot{\vec{x}} \cdot \vec{A} - q\Phi,$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Show that

$$H = \sqrt{(mc)^2 + \left(\vec{p} - \frac{q}{c} \vec{A}\right)^2} + q\Phi.$$

(b) Verify that Hamilton's equations for this system,

$$\dot{x}^k = \frac{\partial H}{\partial p^k}, \quad \dot{p}^k = -\frac{\partial H}{\partial x^k}$$

give the relativistic Lorentz force equation,

$$\frac{d}{dt}(m\gamma \dot{x}^k) = q \left(\left[\frac{\dot{\vec{x}}}{c} \times \vec{B} \right]^k + E^k \right),$$

where

$$E^k = -\frac{\partial \Phi}{\partial x^k} - \frac{1}{c} \frac{\partial A^k}{\partial t},$$

$$B^k = [\nabla \times \vec{A}]^k.$$

4. Consider a particle of mass m that is constrained to move on the surface of a paraboloid whose equation (in cylindrical coordinates) is $r^2 = 4az$. (Note that r is not the overall radius vector.) The particle is subjected to a constant gravitational force $F_z = -mg$.

(a) Show that the Lagrangian is

$$L = \frac{1}{2}m \left[\left(1 + \frac{r^2}{4a^2}\right)\dot{r}^2 + r^2\dot{\theta}^2 \right] - (mg/4a)r^2$$

(b) Find the equation of motion for r .

(c) Show that for a circular orbit ($r = \rho$)

$$l^2 = \frac{m^2g}{2a}\rho^4$$

where l is the angular momentum.

(d) Show that the frequency of small oscillations about the circular orbit is

$$\omega^2 = \frac{2g}{a + \rho^2/4a}$$

Hints:

- Use $r \rightarrow \rho + x$ where $x/\rho \ll 1$.
- Drop terms that contain a product of two or more terms in x and its derivatives.
- Use the relation in (c) to simplify your equation.

5. A uniform rod of length $2L$ and mass M is fixed at the center and free to rotate about the center point. On the rod is a point mass m that is constrained to the rod but otherwise free to move along it. (for the rod: $I = 1/12 ML^2$)

a) For the rod constrained to a vertical plane in a uniform gravitational field, write the Lagrangian and the equations of motion for the mass m .

b) Add the constraint that the rod rotates at a uniform angular velocity. Write the Lagrangian and equations of motions under these conditions.

6. a) Write the rotation matrices that represent rotations about the coordinate axes.

b) Construct, by any means, a rotation matrix that results in the transformation:
 $x \rightarrow y$, $y \rightarrow z$, and $z \rightarrow x$

b) By considering the small angle limit of the matrices in part a), construct the matrices that generate these rotations

c) Explicitly determine the commutation relations of the matrices determined in part b).

2005 Preliminary Examination

Part II: Quantum Mechanics

Time: 3 hours

Date: May 26, 2005

Instructions: Work five (5) of the following six problems. Work each problem on a separate sheet of paper and turn in only the five problems you wish to have graded. Each problem is designed to require approximately 30 minutes to solve. Use your time wisely and be careful to not spend too much time on any one problem. Good Luck!

1. Consider a three-dimensional ket space, and assume that the kets $|1\rangle$, $|2\rangle$ and $|3\rangle$ are orthonormal and form a complete basis. In this basis the operators A and B are represented by

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -a \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -ib & 0 \\ ib & 0 & 0 \\ 0 & 0 & b \end{pmatrix},$$

where a and b are two real constants.

- (a) Does B exhibit a degenerate spectrum?
- (b) Show that A and B commute.
- (c) Find a new set of orthonormal kets which are simultaneous eigenkets of both A and B .
- (d) Specify the eigenvalues of A and B for each of the three new kets. Does your specification of eigenvalues completely characterize each eigenket?
2. Given an atomic system with only two stationary states $|1\rangle$ and $|2\rangle$ and energies E_1 and E_2 where $E_1 < E_2$. Assume that $|1\rangle$ and $|2\rangle$ are orthonormal and that the system is initially in its ground state. At the time $t = 0$, a perturbation H' not depending upon time is switched on. Calculate the probability of finding the system in either state at the time $t > 0$, by assuming that $H'_{11} = 0 = H'_{22}$, where $H'_{ab} \equiv \langle a | H' | b \rangle$, ($a, b = 1, 2$).

3. Consider a physical system with Hamiltonian H . Show that, if $|\Psi(t)\rangle$ is a state of the system and $\langle H \rangle = \langle \Psi(t) | H | \Psi(t) \rangle$ is stationary with respect to infinitesimal variations of $|\Psi(t)\rangle$, then $|\Psi(t)\rangle$ must be an eigenstate of H and determine the respective eigenvalue.

4. The Hamiltonian for an atom can be written as

$$H = H_0 + AL \cdot S$$

where the first term contains the kinetic and electrostatic potential energy terms and the second term is the spin-orbit interaction. The operator A depends only on radial coordinates.

- (a) Using angular momentum commutation rules, show that L_x and S_x (which do commute with H_0) cannot be known exactly because of the spin-orbit interaction. Then show that $J_x = L_x + S_x$ can be known exactly. What other observables involving angular momenta can be known exactly?
- (b) Calculate the spin-orbit energy to first order. (Hint: Use the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$ in your answer.)

5. A particle of mass m is interacting in three dimensions with a spherically symmetric potential

$$V(\mathbf{r}) = -\frac{\hbar^2 \beta}{2ma} \delta(|\mathbf{r}| - a), \quad (1)$$

where β is a dimensionless positive constant. The potential is attractive, and vanishes unless the particle is precisely a distance a from the center.

- (a) The s -wave function for the particle interacting with this potential is proportional to $\sin(kr + \delta)$ for $r > a$. Find an expression for the phase shift δ in terms of k , a , and β .

- (b) Find the total scattering cross section in the limit of small incident energy (but do not set $k = 0$). What is its angular distribution?

- (c) What is the limiting value of the cross section at zero incident energy? Deduce the minimum value of β for which a bound state exists for this potential. Can there be more than one bound state?

6. Consider a physical system whose state space, which is three-dimensional, is spanned by the orthonormal basis formed by the three kets $|u_1\rangle$, $|u_2\rangle$, $|u_3\rangle$. In this basis, the Hamiltonian operator H of the system and the two observables A and B are written

$$H = \hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, A = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, B = b \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where ω_0 , a , and b are positive real constants.

The physical system at time $t = 0$ is in the state $|\psi(t=0)\rangle = (\frac{1}{2})^{1/2} |u_1\rangle + \frac{1}{2} |u_2\rangle + \frac{1}{2} |u_3\rangle$.

- (a) At time $t = 0$ the energy of the system is measured. What values can be found and with what probabilities? Calculate, for the system in the state $|\psi(t=0)\rangle$, the mean value $\langle H \rangle$ and the root-mean-square deviation ΔH .
- (b) Instead of measuring H at time $t = 0$, one measures A . What results can be found and with what probabilities? What is the state vector immediately after each measurement?
- (c) Calculate the state $|\psi(t)\rangle$ of the system at time t .
- (d) Calculate the mean values $\langle A \rangle$ and $\langle B \rangle$ of A and B , respectively, at time t .

2005 Preliminary Examination

Part III: E&M

Time: 3 hours

Date: May 27, 2005

Instructions: Work five (5) of the following six problems. Work each problem on a separate sheet of paper and turn in only the five problems you wish to have graded. Each problem is designed to require approximately 30 minutes to solve. Use your time wisely and be careful to not spend too much time on any one problem. Good Luck!

1. Two plane semi-infinite slabs of the same uniform, isotropic, nonpermeable, lossless dielectric with index of refraction n are parallel and separated by an air gap of width d . A plane electromagnetic wave of frequency ω is incident on the gap from one of the slabs with angle of incidence i . For linear polarization both parallel to and perpendicular to the plane of incidence,

- calculate the ratio of power transmitted into the second slab to the incident power and the ratio of the reflected to incident power;
- for i greater than the critical angle for total internal reflection, sketch the ratio of transmitted power to incident power as a function of d measured in units of wavelength in the gap.

2. A rectangular box occupying the region $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$ has an electrostatic potential $V(x, y, z) = 0$ on the faces $x = 0$, $y = 0$, $z = 0$, and constant potential $V = V_1, V_2, V_3$ on the faces $x = a$, $y = b$, $z = c$ respectively. Starting from Laplace's equation find the potential V everywhere inside the box.

3. A sphere of radius R has a *bound* charge Q distributed uniformly over its surface. The sphere is surrounded by a uniform fluid dielectric medium with fixed dielectric constant ϵ . The fluid also contains a *free* charge density given by

$$\rho(\mathbf{r}) = -k\phi(\mathbf{r}) \quad (1)$$

where $\phi(\mathbf{r})$ is the scalar potential in the fluid and k is a constant.

- Calculate $\phi(\mathbf{r})$ throughout the fluid, with $\phi = 0$ at $r \rightarrow \infty$.
- Find the electric field throughout the fluid.
- Calculate the pressure in the fluid.

4. Consider a region bounded by (1) a semi-infinite conducting plane at $x = 0$ occupying that half of the yz -plane corresponding to positive y ($0 < y \leq \infty$, $-\infty \leq z \leq \infty$); by (2) a similar plane at $x = L$; and by (3) the strip between them in the xz -plane (thus, $0 \leq x \leq L$). Assume the following boundary conditions for the time-independent electric potential $\phi(x,y,z)$:

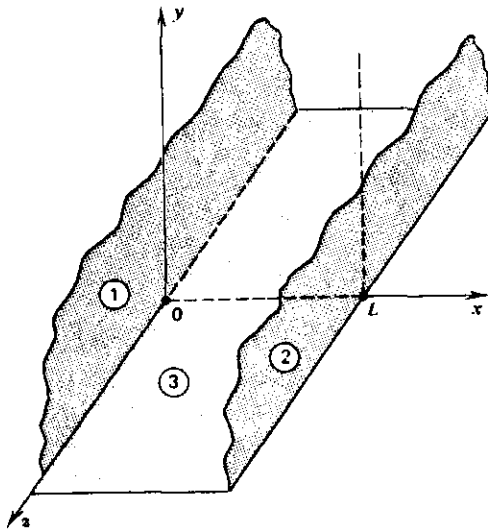
$$\text{At } x = 0, \phi(0,y,z) = 0$$

$$\text{At } x = L, \phi(L,y,z) = 0$$

$$\text{At } y = 0, \phi(x,0,z) = \phi_0, \text{ (a non-zero, positive constant)}$$

$$\text{At } y = \infty, \phi(x,\infty,z) = 0$$

Determine the general solutions for both the scalar potential $\phi(x,y,z)$ and the electric field vector $\vec{E}(x,y,z)$ within the bounded region.



5. A uniformly charged solid sphere of radius R carries a total charge Q , and is set spinning with angular velocity ω about the z axis.

(a) Show that the magnetic dipole moment of the sphere is

$$\vec{m} = \frac{1}{5} Q \omega R^2 \hat{z} .$$

(b) Find the vector potential at a point (r, θ) where $r \gg R$.

(c) What is the magnetic field at this point ?

6. (a) Consider the case of propagating, degenerate TE waveguide modes ($E_z=0$), subject to finite conductivity boundary conditions. Given the TE boundary condition (" \mathbf{n} " represents an outward normal, " \mathcal{L} " a counterclockwise derivative along the surface; $\psi = H_z$),

$$\frac{\partial \psi}{\partial \mathbf{n}} \Big|_s = -f^{\text{TE}} \left(\psi - \frac{k^2}{\gamma_0^4} \frac{\partial^2 \psi}{\partial \mathcal{L}^2} \right) \Big|_s.$$

($k^2 = \frac{\mu\epsilon}{c^2} \omega^2 - \gamma_0^2$) Show that, assuming n fold degeneracy and that the degenerate TE modes are orthogonal,

$$\sum_{i=1}^n [(\gamma^2 - \gamma_0^2) N_i \delta_{ji} + \Delta_{ji}] a_i = 0,$$

$$N_i = \int da |\psi_0^{(i)}|^2,$$

$$\Delta_{ji} = - \oint d\mathcal{L} f^{\text{TE}} \left[\psi_0^{(j)*} \psi_0^{(i)} + \frac{k^2}{\gamma_0^4} \frac{\partial \psi_0^{(j)*}}{\partial \mathcal{L}} \frac{\partial \psi_0^{(i)}}{\partial \mathcal{L}} \right],$$

where $\psi_0^{(i)}$ are the original degenerate eigenmodes:

$$(\nabla_{\perp}^2 + \gamma_0^2) \psi_0^{(i)} = 0, \quad \frac{\partial \psi_0^{(i)}}{\partial \mathbf{n}} \Big|_s = 0.$$

(b) Slight changes in conductivity on the waveguide boundaries can split the eigenmodes of a cylindrical waveguide. This can be described by the theory in part (a), where the quantity $f^{\text{TE}} \rightarrow f^{\text{TE}}(\mathcal{L})$ can be regarded as a surface variable and kept inside the integral. The 2 lowest TE cylindrical modes are specified by (assume H_0 is chosen such that $N_i = 1$)

$$\psi_0^{(j)} = H_0 e^{z i \phi} J_1(\gamma \rho), \quad \text{root condition: } J_1'(x) = 0, \quad x = \gamma R.$$

(lowest $x = 1.84118$.) Taking (due to temperature or surface variations)

$$f^{\text{TE}}(\phi) = f_1 + f_2 \sin(2\phi), \quad (|g_0| \ll |f_0|)$$

($f_{1,2}$ constants) use the theory in (a) to find the splitting of the lowest modes.

2005 Preliminary Examination

Part IV: General, Mathematical & Statistical Physics

Time: 3 hours

Date: May 27, 2005

Instructions: Work five (5) of the following six problems. Work each problem on a separate sheet of paper and turn in only the five problems you wish to have graded. Each problem is designed to require approximately 30 minutes to solve. Use your time wisely and be careful to not spend too much time on any one problem. Good Luck!

1. (a) Hall Effect experiments show that in some solids the charge carriers have negative charge (ascribed to electrons) and in others, positive charge (ascribed to electron holes). Prove that fermion holes are also fermions.

(b) Consider N fermions distributed over a finite number $d = 2N$ of single-particle states. What property of the single-particle spectrum will be necessary and sufficient to ensure that the chemical potential $\mu = 0$ for all values of T ? What is the physical explanation of this result?

2 The one-dimensional neutron diffusion equation with a (plane) source is

$$-D \frac{d^2 \varphi(x)}{dx^2} + K^2 D \varphi(x) = Q \delta(x)$$

on $-\infty < x < \infty$, where $\varphi(x)$ is the neutron flux, $Q\delta(x)$ is the (plane) source at $x = 0$, and D and K^2 are constants.

- Apply a Fourier transform. Solve the equation in transform space. Transform your solution back into x -space by performing an integral in the complex plane. Show all steps.
- The ODE is similar to a Green Function differential equation. Use a Green Function technique (but not a Fourier transform) to solve the equation.

3. A quantum mechanical analysis of the Stark effect in parabolic coordinates leads to the differential equation

$$\frac{d}{d\xi} \left(\xi \frac{du}{d\xi} \right) + \left(\frac{1}{2} E \xi + \alpha - \frac{m^2}{4\xi} - \frac{1}{4} F \xi^2 \right) u = 0,$$

where α is a separation constant, E is the total energy, and F is related to the electric field strength.

- Using the upper root of the indicial equation, develop a power series solution about $\xi = 0$. Evaluate the first three coefficients in terms of a_0 .
- Explain how you might go about finding a second solution of the differential equation. (Note: You do not need to find a second solution; just explain the steps to be taken.)

4. For ideal quantum gas, occupancy $\langle n_i \rangle$ is given in terms of $Z(\alpha, \beta)$ by $\langle n_i \rangle = -(1/\beta) \partial / \partial \epsilon_i (\ln Z(\alpha, \beta))$. $Z(\alpha, \beta)$ is the grand partition function of the system. Derive the corresponding result for the "multiple occupancy" $\langle n_s n_t n_u \dots \rangle$, where s, t, u are not necessarily distinct.

5. An infinite, uniform plasma with fixed ions has an electron distribution function composed of a Maxwellian distribution of plasma electrons with density n_p and temperature T_p (at rest in the laboratory) and a Maxwellian distribution of beam electrons with density n_b and temperature T_b , centered at $\mathbf{v} = V \mathbf{x}$. If n_b is infinitesimally small, plasma oscillations traveling in the x direction will be damped. (This is known as Landau-damping.) If n_b is large, a two-stream instability will develop. The critical n_b at which this instability occurs can be determined by setting the slope of the total distribution function equal to zero. To obtain a simple approximation for this value, follow the prescription given below.

(a) Find expressions for $f_p(v)$ and $f_b(v)$, in terms of $v = v_x$, $a^2 = 2KT_p/m$ and $b^2 = 2KT_b/m$.

(b) Assume that the phase velocity v_ϕ will be the value of v at which $f_b(v)$ has the largest positive slope. Find v_ϕ and $f'_b(v_\phi)$.

(c) Find $f'_p(v_\phi)$ and set $f'_p(v_\phi) + f'_b(v_\phi) = 0$.

(d) For $V \gg b$, find an approximate value for the critical beam density (n_b/n_p) in terms of T_b, T_p, V and a .

6. Calculus of Variation: Find the stationary function $y(x)$, (with associated condition on u) resulting from the problem:

$$\delta \int_0^u [y'^2 + 4(y - u)] dx = 0$$

where $y' = dy/dx$ and where u is not preassigned. The boundary conditions are:

$$y(0) = 2, \quad y(u) = u^2$$

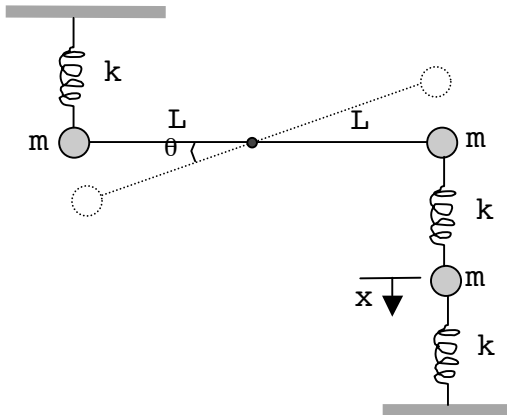
Ph.D. Preliminary Examination, 2006

Mon., May 22, 2006

Part I: Classical Mechanics (9:00 a.m. to 12:00 p.m.)

1. A massless stiff rod of length $2L$ is pivoted at its center and constrained to move in a vertical plane by springs and masses at each end as shown in the accompanying figure.

- Set up the Lagrangian $L(x, \dot{x}, \theta, \dot{\theta})$ and determine the two equations of motion.
- Calculate the eigenfrequencies of the motion.
- Find the normalized eigenvector corresponding to the largest eigenfrequency.



2. A particle of mass m moves under the influence of the central force

$$\mathbf{F} = -c^2 \frac{\mathbf{r}}{r^{5/2}}.$$

- Calculate the potential energy and the (fictitious) effective potential.
- Find the radius of any circular orbit in terms of the angular momentum.

3. A pendulum consists of two masses connected by a very light rigid rod, as shown. The pendulum is free to oscillate in the vertical plane about a horizontal axis

located a distance a from m_a at a distance b from m_b .

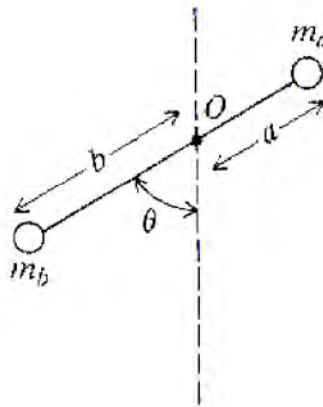
a. Calculate the moment of inertia of the system about O . Find the location of the center of mass.

b. Set up the Lagrangian and determine the equation of motion for the system.

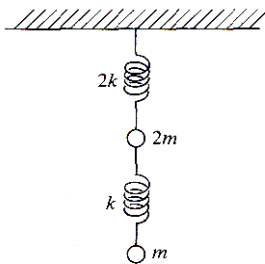
c. Take $b > a$ and determine the frequency of oscillation for small angles of displacement from the vertical.

d. Derive an exact expression for the period of the pendulum ($|\theta_{\max}| < \pi$).

e. Find the minimum angular velocity which must be given to the system (starting at equilibrium) if it is to continue in rotation instead of oscillating.



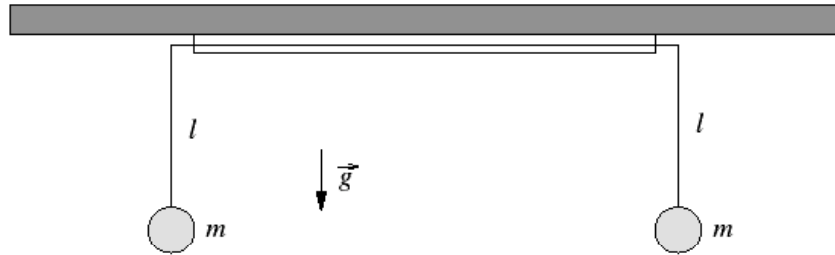
4. A mass $2m$ is suspended from a fixed support by a spring with spring constant $2k$. A second mass m is suspended from the first mass by a spring of constant k . Find the equations of motion for this coupled system and determine the frequencies of oscillation of normal modes. Neglect the masses of the springs. The equilibrium lengths of the springs without gravity are both zero. You must include gravity to determine the equilibrium positions of the masses.



5.

A long massless rod is mounted inside a slightly shorter tubular bushing, and the bushing is rigidly mounted to the ceiling. The rod is elastic and follows the formula $\tau = -\kappa\phi$ where τ is the torque and ϕ is the angular twist in the rod.

The rod extends slightly from each end of the bushing and to each end is welded a rigid, massless rod of length l with a mass m attached at the lower end as shown in the figure. The pendula are constrained to move only in a plane perpendicular to the paper.



Draw a well-labeled picture to define your choice of coordinates. Then, in the small angle approximation, determine the Lagrangian for this system.

6.

A small bead of mass m is constrained to move along a sinusoidally shaped wire, on which it slides without friction. The equation describing the wire is $y = b \sin(qx)$, where x and y are standard Cartesian coordinates. The bead is also attached by a Hooke's law spring to a fixed object at $(x, y) = (0, d)$. The spring has spring constant k and equilibrium length L .

- (a) Find a Hamiltonian for this system, using x as the only generalized coordinate.
- (b) The bead is held stationary at $x = 4\pi/q$ and then released. Find the speed of the bead when it passes through the origin. (Assume the parameters are chosen such that the bead does eventually reach the origin.)

Ph.D. Preliminary Examination, 2006

Mon., May 22, 2006

Part II: Quantum Mechanics (1:00 p.m. to 4:00 p.m.)

1. Assume that $\Psi_1(x, t)$ and $\Psi_2(x, t)$ are two solutions of the one-dimensional Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} \Psi(x, t) = \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right) \Psi(x, t).$$

(a) Show that the expression,

$$\int_{-\infty}^{\infty} \Psi_1^*(x, t) \Psi_2(x, t) dx,$$

is time-independent.

(b) Assume that a particle is moving under the potential,

$$V(x) = \begin{cases} 0, & |x| < a/2, \\ V_0, & |x| \geq a/2, \end{cases}$$

where $V_0 > 0$. Find the condition for the existence of the first bound state ($E \simeq V_0$).

(c) Consider the one-dimensional scattering problem, with

$$V(x) = \begin{cases} 0, & x < 0, \\ V_0, & x \geq 0, \end{cases}$$

where $V_0 > E > 0$. Assuming that the incoming wave is given by $\Psi(x, t) = e^{i(kx - \omega t)}$, $E \equiv \hbar^2 k^2 / 2m$, find the reflection probability coefficient.

2. The orbital angular momentum \mathbf{L} is defined by $\mathbf{L} \equiv \mathbf{r} \times \mathbf{p}$.

(a) Using the fundamental relations, $[x_i, p_j] = i\hbar \delta_{ij}$, prove that $[L_i, L_j] = i\hbar \epsilon_{ijk} L_k$.

(b) Assuming that $|l, m_z\rangle$ is an eigenket of L_z , and using the results obtained in (a), show that the corresponding expectation values of L_x and L_y are all zero.

(c) Calculate the expectation values of L_x^2 and L_y^2 for the state $|l, m_z\rangle$, and then verify the uncertainty principle.

(d) For the eigenket $|l = 1, m_z = 0\rangle$ of L_z , what are the possible measured values of L_x and their corresponding probabilities?

3. Consider a particle of mass m submitted to the potential:

$$V(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq a \\ +\infty, & \text{if } x < 0 \text{ or } x > a. \end{cases}$$

$|\varphi_n\rangle$ are the orthonormal eigenstates of the Hamiltonian H of the system, and their respective eigenvalues are

$$E_n = n^2 \pi^2 \hbar^2 / (2ma^2).$$

The state of the particle at the instant $t = 0$ is

$$|\psi(t = 0)\rangle = a_1 |\varphi_1\rangle + a_2 |\varphi_2\rangle + a_3 |\varphi_3\rangle + a_4 |\varphi_4\rangle,$$

where the a_i are (possibly complex) normalized coefficients, i.e., $\sum_i |a_i|^2 = 1$.

(a) What is the probability, when the energy of the particle in the state $|\psi(t=0)\rangle$ is measured, of finding a value of the energy smaller than $3\pi^2\hbar^2/(ma^2)$?

(b) What is the mean value of the energy, $\langle E \rangle$, of the particle in the state $|\psi(t=0)\rangle$? (Write the answer in terms of $E_1 = \pi^2\hbar^2/(2ma^2)$.)

(c) Write out the state $|\psi(t)\rangle$ at a time $t > 0$. If no energy measurement is made at $t = 0$, do the results found in (a) and in (b) at the instant $t = 0$ remain valid at an arbitrary time $t > 0$? Why or why not?

(d) When the energy is measured at $t = 0$, the result is $8\hbar^2/(ma^2)$. What will be the probability of the same value of energy being measured at a later time t ? After the energy is measured at $t = 0$, what is the probability of the particle being in the $n = 2$ state at a later time t ?

4. (a) In quantum mechanics, when does an operator O correspond to a constant of motion?

(b) The Hamiltonian for a certain particle of mass m moving in a three-dimensional space is

$$H = (p_x^2 + p_y^2 + p_z^2)/(2m) + \lambda x,$$

where λ is a non-zero constant. Which of the following operators x, y, z, p_x, p_y, p_z are constants of motion? Which are not constants of motion? Justify your answers in terms of your discussion in (a).

5. Consider the scattering of spinless particles of mass m and charge e in the screened Coulomb potential

$$V(\mathbf{r}) = \frac{Ze}{r} e^{-r/a}.$$

(a) [40%] Find the transition amplitude (matrix element) for a plane wave of these particles with non-relativistic momentum \mathbf{p} scattering elastically from this potential into a plane wave with momentum \mathbf{p}' . Express your result in terms of $q = |\mathbf{p} - \mathbf{p}'|/\hbar$.

(b) [30%] Describe an approximation that can be used to calculate the differential cross section for fast (but non-relativistic) particles. Obtain the differential cross-section $d\sigma/d\Omega$ using this approximation. (Do not be concerned about overall dimensionless constants.) What is its range of validity?

(c) [30%] Describe an approximation that can be used to calculate the differential cross section for slow particles, with no assumptions about the strength of the potential. What are the angular distribution and energy spectrum of the particles in this case? (Do not be concerned with the normalization.) Is there a limit in which the results of (b) remain applicable for low-energy scattering? If so, what is it, and are the results of (b) consistent with your general expectations for the scattering of slow particles?

6.

Consider a quantum system for which the exact Hamiltonian is H . Assume the quantum system is of bounded extent so that it is known rigorously that the eigenstates of H , $\{|\psi_n\rangle\}$, are complete.

(a). Show that, if $|\psi_n\rangle$ and $|\psi_m\rangle$ are two eigenstates of H with eigenvalues E_n and E_m respectively with $E_n \neq E_m$, then $\langle \psi_n | \psi_m \rangle = 0$.

(b). Suppose $E_n = E_m$, $n \neq m$. Can we still have $\langle \psi_n | \psi_m \rangle = 0$? Explain your answer.

(c). The problem $H|\psi\rangle = E|\psi\rangle$ is very complicated but it is suggested that we use a trial wave function $|\psi_{\text{trial}}\rangle$ for $|\psi\rangle$ and approximate E by

$$E = \frac{\langle \psi_{\text{trial}} | H | \psi_{\text{trial}} \rangle}{\langle \psi_{\text{trial}} | \psi_{\text{trial}} \rangle}.$$

Show that $E \geq E_0$ where E_0 is the lowest eigenvalue of H .

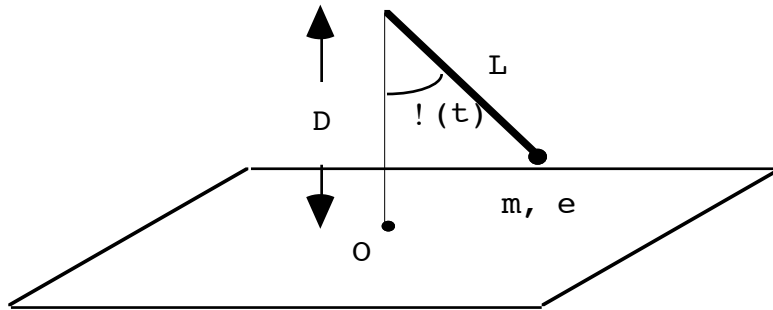
Ph.D. Preliminary Examination, 2006

Tues., May 23, 2006

Part III: Electricity & Magnetism (9:00 a.m. to 12:00 p.m.)

1. Consider that all space is filled with a dielectric of dielectric constant ϵ . The electric field \vec{E}_0 is non-zero and uniform throughout all space. A small spherical hole of radius a is now cut into the dielectric. Find the electrostatic potential ϕ and the electric field \vec{E} inside and outside the hole.

2. A particle of charge e is attached to a massless string of length L and oscillates above a semi-infinite conducting plane that fills the entire region below the origin, O . The distance from the point of attachment to the plane is D . (Gravity is *not* operating in this problem and $D > L$.)



By computing the potential energy of the system to lowest order in the angle $\theta(t)$ (or other means), find the frequency of small oscillations of the charge.

3. For a harmonic current density source of the form

$$\vec{J}'(\vec{x}', t) = \vec{J}(\vec{x}') e^{i\omega t},$$

one may show that

$$\left. \frac{dP}{d\Omega} \right|_{\text{avg.}} = \frac{1}{T} \int_0^T dt \frac{dP}{d\Omega}(t) = \frac{\mu_0 \omega^2}{32\pi^2 c} \left| \hat{n} \int d^3x' \vec{J}(\vec{x}') e^{-ik\hat{n}\cdot\vec{x}'} \right|^2,$$

for the average angular radiated power, where $k = \frac{\omega}{c}$.

(a) Now given

$$\vec{J}'(\vec{x}', t) = I\delta(x')\delta(y')\hat{k},$$

(I is a constant current) which represents a coherent harmonic line source located along $-d/2 < z' < d/2$, show that the average angular radiated power is

$$\left. \frac{dP}{d\Omega} \right|_{\text{avg.}} = \frac{\mu_0 c I^2}{8\pi^2} \frac{\sin^2\left(\frac{kd}{2} \cos\theta\right)}{\cos^2\theta} \sin^2\theta.$$

(b) Treat the same problem in electric dipole approximation using the formula

$$\left. \frac{dP}{d\Omega} \right|_{\text{avg.}} = \frac{\mu_0 c I^2}{8\pi^2} |\hat{n} \cdot \vec{p}|^2,$$

where \vec{p} is the electric dipole moment constant for this harmonic source. Reconcile your answer with the (a) part.

4. a. An inverted hemispherical bowl of radius R carries a uniform surface charge density σ . Find the potential difference between the "north pole" and the center.
- b. An infinitely long cylinder, of radius R , carries a "frozen-in" magnetization, $\mathbf{M} = kr^2 \mathbf{z}$, where k is a constant, r is the distance from the axis and \mathbf{z} is the unit vector in the direction of cylinder axis. Locate the all bound currents and calculate the magnetic field at a point, (a) inside, (b) outside the cylinder.

5. An infinitely long solenoid of radius a , with n turns per unit length, carries a current I_s . Coaxial with the solenoid, at radius $b \gg a$, is a circular ring of wire with resistance R . When the current in the solenoid is gradually decreased, a current I_r is induced in the ring.

(a) Find the I_r in terms of dI_s/dt .

(b) Find the electric field at the surface of the solenoid due to the changing flux in the solenoid.

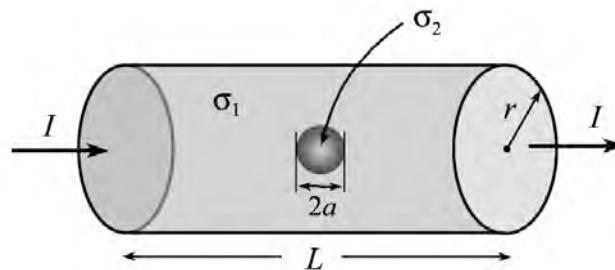
(c) Show that the magnetic field on the axis of the solenoid due to the current in the ring is given by

$$\vec{B} = \frac{\mu_0 I r}{2} \frac{b^2}{(b^2 + z^2)^{3/2}} \hat{z},$$

where z is the distance from the center of the ring. (This is essentially the B field on the surface of the solenoid.)

(d) The power ($I^2 R$) delivered to the ring must come from the solenoid. Confirm this by calculating the Poynting vector at a point on the surface of the solenoid and integrating it over the entire surface.

6. A resistor is in the form of a solid cylinder of radius r , length L , and is made of a material with conductivity σ_1 . At the center of the resistor is a defect consisting of a small sphere of radius a inside which the conductivity is σ_2 . The input and output current I is distributed uniformly across the flat ends of the resistor.



(a) [20%] What is the resistance R_0 of the resistor without the defect (*i.e.* with $\sigma_2 = \sigma_1$)?

(b) [40%] Estimate the relative change $\Delta R/R = (R - R_0)/R_0$ in the resistance due to the defect to first order in the relative change in the conductivity of the defect, $\Delta\sigma/\sigma = (\sigma_2 - \sigma_1)/\sigma_1$. (Make any assumptions needed to arrive at a quick but reasonable estimate.)

(c) [40%] Suppose $L \rightarrow \infty$ and $R \rightarrow \infty$, but a uniform current density J_0 continues to flow across the ends of the resistor. Calculate (exactly) the current density inside the spherical defect in this limit.

Ph.D. Preliminary Examination, 2006

Tues., May 23, 2006

**Part IV: Statistical Mechanics and Mathematical Physics
(1:00 p.m. to 4:00 p.m.)**

1. In the Debye theory of solids, we have both for transverse and longitudinal waves that the frequency ω and wave number k are related by $\omega = ck$. The theory extends easily to the propagation of spin waves (“magnons”), excitations in which the direction of magnetization in a ferromagnetic material propagates as a wave. In this case $\omega \propto k^2$. How does the spin-wave energy U vary with T at low T ?

2. For a monatomic spin-less ideal classical gas, the chemical potential μ is given by $\mu = \beta^{-1} \ln(\lambda_T^{-2}/v)$ where λ_T is thermal wavelength and v is volume per particle. Drive the chemical potential for the case where the particles have spin J .

3. Due to its spin, the electron possesses a magnetic moment μ_B . Therefore, in the presence of a magnetic field B , an electron with momentum p has an energy dependent upon whether its magnetic moment is parallel or antiparallel to the magnetic field.

a. Treating the conduction electrons in a metal as a free electron gas, obtain an expression (involving integrals over Fermi-Dirac distribution functions) for the magnetization due to the magnetic moments of the conduction electrons, when placed in a magnetic field.

b. Evaluate this expression at absolute zero.

c. Plot the two density of states that enter into your calculation above and use this graph to discuss Pauli spin paramagnetism at 0 K.

d. Use your results to argue that only a fraction T/T_F of the conduction electrons contribute to the magnetization. Use this to explain why an external magnetic field can lead to the diamagnetic effect first calculated by Landau.

4. According to the Rayleigh-Jeans law, the mean energy U per normal mode of black-body radiation at a temperature T is

$$U = kT.$$

According to Wien’s law

$$U = \hbar\omega e^{-\beta\hbar\omega}$$

- When is the Rayleigh-Jeans law valid? When is Wien's law valid?
- Calculate $d(1/T)/dU$ for both cases.
- Invent an interpolation formula for $d(1/T)/dU$ between these two limits. Once you have this formula, integrate it to obtain an expression for U that is correct for any frequency. (Hint: If you choose the correct interpolation formula, this procedure should result in Planck's law.)

5. The three-dimensional Dirac delta function can be represented as an integral of the form:

$$\delta(\vec{r}) = \int \frac{d\vec{k}}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{r}} D(\vec{k})$$

where $D(\vec{k})$ is the Fourier transform of the delta function.

- Find $D(\vec{k})$.
- From the integral representation of the delta function and the fact that the Coulomb potential $\phi(\vec{r}) = e/r$ satisfies Poisson's equation,

$$-\nabla^2 \phi(\vec{r}) = -4\pi e \delta(\vec{r})$$

show that the electron-electron pair potential, $V(\vec{r}) = -e\phi(\vec{r}) = e^2/r$ can be written in the form

$$V(\vec{r}) = \int \frac{d\vec{k}}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{r}} V(\vec{k})$$

where

$$V(\vec{k}) = \frac{4\pi e^2}{(2\pi)^{3/2} k^2}$$

- Prove that the Fourier transform of the screened Coulomb interaction $V_s(r) = (e^2/r)e^{-k_0 r}$ is

$$V_s(\vec{k}) = \frac{4\pi e^2}{(2\pi)^{3/2} (k^2 + k_0^2)}$$

by substituting the $V_s(\vec{k})$ expression into the Fourier integral

$$V_s(\vec{r}) = \int \frac{d\vec{k}}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{r}} V_s(\vec{k}).$$

and evaluating the integral (without integral tables) in spherical coordinates. Choose a contour in the complex plane to perform the radial integration.

(**Note:** The integrand has an angular dependence that must be dealt with first.)

6. A string is clamped at both ends, $x = 0$ and $x = L$. Assuming small amplitude vibrations, we find that the amplitude $u(x,t)$ satisfies the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

where c is the wave velocity.

a. Find the normal modes of vibration. Give the general solution for an arbitrary initial state ($t = 0$) of vibration.

b. Consider the case when the string is set in motion by a sharp blow at $x = a$. Thus, $u(x,0) = 0$ and $\frac{\partial u}{\partial t}(x,0) = Lv_0 \delta(x-a)$.

Solve the wave equation subject to these initial conditions.

c. Find the transverse velocity of the string as a function of time.

d. The Green's function $G(x,x';k)$ is often used when several vibrating string problems are to be solved. Find the

Green's function for the operator $\frac{\partial^2}{\partial x^2} + k^2$ subject to the same boundary conditions at $x = 0$ and $x = L$.

Problem 5

Use Green's function to solve the following initial ordinary differential equation

$$x'' + 9x = h(t), \quad x(0) = 1, \quad x'(0) = 1,$$

where $h(t) = 0$ for $t < \frac{\pi}{2}$, and $h(t) = \sin(t)$ for $t > \frac{\pi}{2}$.

Problem 6

a) Determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$$

b) Using the calculus of residues, evaluate the integral,

$$\int_{-\infty}^{\infty} \frac{\cos(mx)}{(x+a)^2 + b^2} dx$$

where a is real, b is real and positive, and m is an integer.