Baylor University Physics Ph.D. Program
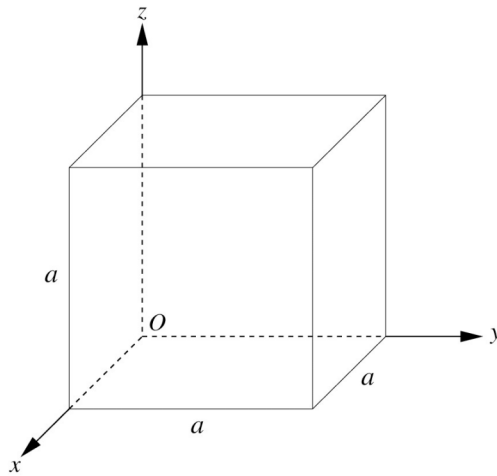## Preliminary Exam 2007

Part I: Classical Mechanics
Friday, May 25, 2007, 9:00 am

Problem 1.

(a) Show that the inertia tensor for a uniform solid cube of mass M and side a rotating about its corner is

$$\mathbf{I} = \frac{Ma^2}{12}\begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}$$

Use axes parallel to the cube's edges with the origin at O, as shown in the figure:

The solid cube is free to rotate about O.

(b) Diagonalize the inertia tensor and find the principal moments of inertia and the principal axes.

Problem 2.

Consider a particle of mass m orbiting in a central force with $V = kr^\alpha$ where α is a real number and $k\alpha > 0$.

(a) Explain what the condition $k\alpha > 0$ tells us about the force. Sketch the effective potential energy for the cases where $\alpha = 2, -1$ and -3.

(b) Find the radius at which the particle (with given angular momentum l) can orbit at a fixed radius. For what values of α is this circular orbit stable?

Problem 3.

Particle 1 of rest mass m_0 moves along the x axis at relativistic velocity v_0 in the laboratory frame, colliding elastically with particle 2, also of mass m_0, initially at rest. After the collision, particles 1 and 2 are observed to move at symmetric angles θ and $-\theta$, respectively, measured from the $+x$ axis.

(a) Using the relativistic co-linear velocity addition formula

$$v_{12} = \frac{v_1 + v_2}{1 + \dfrac{v_1 v_2}{c^2}},$$

or other means, find the velocity of the center-of-momentum (also called center of mass or CM) frame of the two particles with respect to the laboratory frame.

(b) Find the magnitudes of the CM momentum vectors of each particle before and after the collision.

(c) Find the x and y components of the momenta of these two particles after the collision in the laboratory frame, where y labels the transverse direction.

Problem 4.

A particle of mass m moves in a central force field given by the Yukawa potential

$$V(r) = - (\mu/r)\, e^{-\alpha r}$$

where μ and α are positive constants.

(a) After considering the symmetry of the problem, first write down the corresponding Lagrangian, and then find the one-dimensional problem equivalent to its motion.

(b) When are circular orbits possible?

(c) Find the period of small radial oscillations about the circular motion.

Problem 5.

Solve the differential equation of motion of the damped harmonic oscillator driven by an exponentially decreasing harmonic force
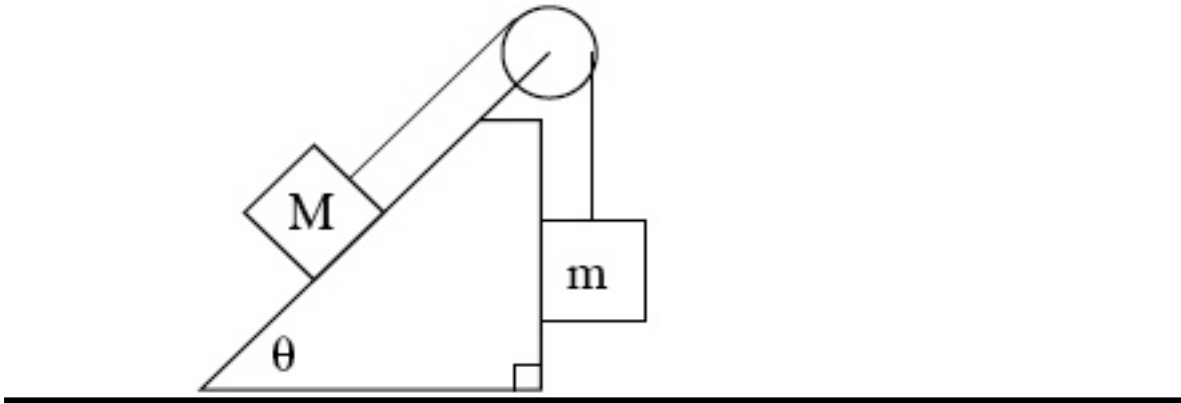
$$F_{ext}(t) = F_0 \exp[-\alpha t] \cos \omega t$$

Hint: $\exp[-\alpha t] \cos \omega t = \text{Re} (\exp[-\alpha t + i \omega t]) = \text{Re} (\exp[\gamma t])$, where $\gamma = -\alpha + i \omega$.
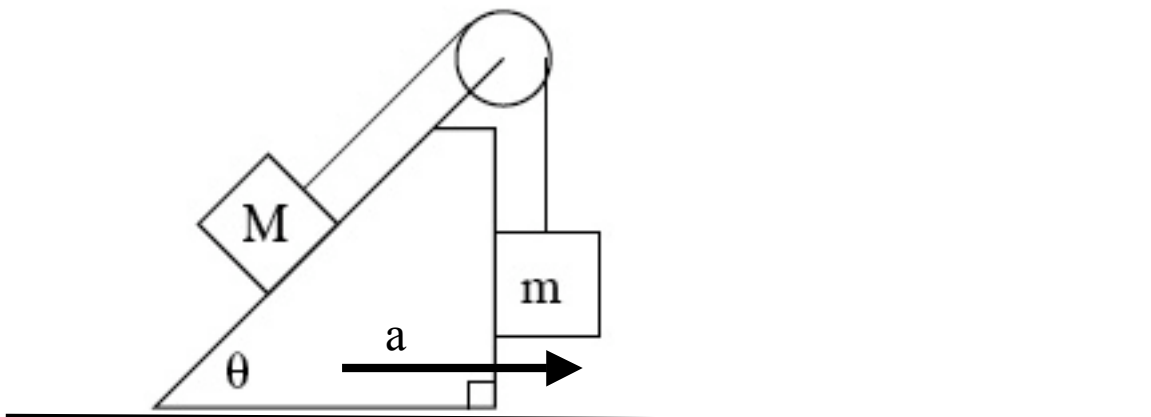
Assume a solution of the form: $A \exp[\gamma t - i\phi]$.

Problem 6.

Consider a system, as shown below, in which a mass M lies on a plane inclined at an angle θ with respect to horizontal. It is attached to a block of mass m by a massless rope that passes over a massless, frictionless pulley. The coefficient of static friction between the mass M and the inclined plane is μ. The vertical side next to mass m is frictionless.

(a) Under what conditions do the blocks remain stationary? (The inclined plane itself is fixed in place.)

(b) The entire system is now accelerated to the right with an acceleration a. In this setting, under what conditions do the blocks remain stationary?

Baylor University Physics Ph.D. Program
Preliminary Exam 2007

Part II: Quantum Mechanics
Friday, May 25, 2007, 1:30 pm

Problem 1.

A 1-D harmonic oscillator at time $t = 0$ is in a state Ψ which is a superposition,

$$\Psi(x, t = 0) = \tfrac{1}{2}\, u_0 + (1/\sqrt{2}\,)\, u_1 + \tfrac{1}{2}\, u_2,$$

where u_0, u_1, and u_2 are the normalized ground, 1st excited, and 2nd excited harmonic oscillator eigenstates, respectively.

(a) Find $\langle x^2 \rangle$ for all later times.

(b) Find the time average of $\langle x^2 \rangle$.

Recall that

$$x = [\hbar/(2m\omega)]^{1/2} (a + a^\dagger),$$

and

$$p = i[m\hbar\omega/2]^{1/2} (a^\dagger - a),$$

where a and $a^\dagger$ are the lowering and raising operators, respectively.

Problem 2.

A particle of mass m moves in one dimension with a potential $V(x) = k|x|$. Use the variational principle with trial wave function $\Phi = \exp(-\alpha x^2)$ to estimate the energy E_0 of the ground state.
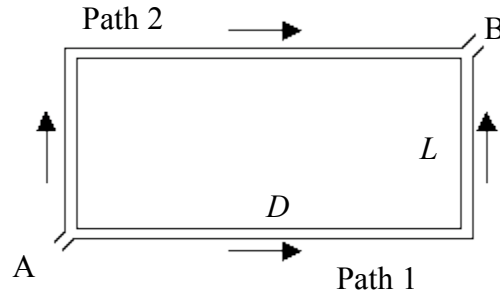
Problem 3.

(a) Write an explicit completely symmetric state of a 3-particle system with correct normalization, with each particle capable of being in the distinct energy states α, β, or γ.

(b) Repeat part (a) for the case of $\alpha \neq \beta = \gamma$.

(c) Repeat part (a) for an explicit antisymmetric state.

(d) Repeat part (b) for an explicit antisymmetric state.

(e) Two indistinguishable bosons with mass m are confined to an infinite square well with width a. Write their generic wavefunctions and energy eigenvalues for their bound states.

Problem 4.

A beam of non-relativistic neutrons can move from point A to point B through an apparatus along two different paths as shown in the figure below. The apparatus is in a vertical plane, so that a neutron feels a downward force *mg* while traveling through the apparatus. As the apparatus is tilted into the horizontal plane, so that the vertical height *H* of Path 2 above Path 1 decreases from *L* to 0, an alternating series of intensity maxima and minima are observed in the neutron beam at point B. (If the apparatus is tilted at an angle θ with respect to vertical, $H = L \cos\theta$.)

Explain this phenomenon in the WKB approximation using the wavefunction expression

$$\psi(x) = \psi_0(k,x)\exp\left(i\int_0^x k(x')dx'\right).$$

Assume the prefactor $\psi_0(k,x)$ is approximately a constant and

$$k(x) = \frac{1}{\hbar}\sqrt{2m(E - V(x))}.$$

Calculate the height difference ΔH between intensity maxima as a function of the energy E of the neutron beam, for E >> mgL. You may treat the propagation as one-dimensional.

An alternate solution using de Broglie wavelengths and conservation of energy may also be made.

Problem 5.

A simple example of a solvable two-body problem is the case of two spin ½ particles interacting only through Hooke's law forces:

$$H = -\frac{\hbar^2}{2m}\left(\nabla_1^2 + \nabla_2^2\right) + \frac{1}{2}m\omega^2\left(\mathbf{r}_1^2 + \mathbf{r}_2^2\right) + \frac{1}{2}\kappa|\mathbf{r}_1 - \mathbf{r}_2|^2$$

(a) Show that the change of variables from $\mathbf{r}_1$, $\mathbf{r}_2$, to

$$\mathbf{R} \equiv \frac{1}{\sqrt{2}}(\mathbf{r}_1 + \mathbf{r}_2); \quad \mathbf{r} \equiv \frac{1}{\sqrt{2}}(\mathbf{r}_1 - \mathbf{r}_2)$$

turns the Hamiltonian into two independent three-dimensional harmonic oscillator problems.

(b) From your knowledge of the one-dimensional harmonic oscillator, write down the exact ground state energy for this system.

(c) If κ is sufficiently small, the third term in the Hamiltonian may be viewed as a perturbation. The first two terms then make up the unperturbed Hamiltonian, which again separates into two harmonic oscillators. Using perturbation theory, calculate the ground state energy of the system correct to first-order.

Hint: For a 3-D oscillator, the ground state is $\psi_0(\mathbf{r}) = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-m\omega r^2/2\hbar}$.

Also: $\displaystyle\int_0^\infty x^{2n} e^{-ax^2}\,dx = \frac{1\cdot 3\cdot 5\cdot\ldots(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$

Problem 6.

The spin angular momentum operator is denoted by the symbol $\hat{S}$.

(a) Write down expressions for the following three commutation relations:

$$[\hat{S}_x,\hat{S}_y],\quad [\hat{S}_y,\hat{S}_z],\quad [\hat{S}_z,\hat{S}_x],$$

where $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$ are the Cartesian components of $\hat{S}$.

(b) Use the Pauli spin matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

to confirm these three commutation relations.

(c) What is the spin polarization of a beam of electrons described by the density operator

$$\hat{\rho} = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \; ?$$

(d) You make a measurement of the sum of the x and y components of the spin of an electron. What are the possible results of this experiment? After this measurement, you measure the z component of the spin. What are the respective probabilities of obtaining the values $\pm \tfrac{1}{2}\hbar$?

Baylor University Physics Ph.D. Program
Preliminary Exam 2007

Part III: Electricity and Magnetism
Saturday, May 26, 2007, 9:00 am

Problem 1.

(a) If Φ is the electrostatic potential due to a volume charge density ρ within a volume V and a surface charge density σ on the conducting surface S bounding the volume V, and Φ' is the potential due to another charge distribution ρ' and σ', prove Green's reciprocation theorem:

$$\int_V \rho\,\Phi'\,d^3x + \int_S \sigma\,\Phi'\,da \ = \ \int_V \rho'\,\Phi\,d^3x + \int_S \sigma'\,\Phi\,da.$$

(b) Two infinite grounded parallel conducting planes are located at x = 0 and x = d. A point charge, q, is placed between the planes at position x, where 0 < x < d. Using Green's reciprocation theorem with a known comparison problem with the same geometry, find the induced charges on each of the planes.

Problem 2.

(a) Using the fact that the scalar potential, $\Phi(\vec{x})$, satisfies the Laplace equation in charge free space, show that a grounded spherical perfect conductor of radius "a" (zero charge) placed in a uniform electric field, $\vec{E}$, acquires a electric dipole moment,

$$\Phi \equiv \frac{1}{4\pi\varepsilon_0} \frac{\vec{p} \circ \vec{x}}{r^3},$$

where

$$\vec{p} = 4\pi\varepsilon_0 \vec{E} a^3.$$

(b) Show that the same perfect conductor ($\mu = 0$ inside) placed in a uniform magnetic field, $\vec{B}$, acquires a magnetic dipole moment defined by the magnetic scalar potential,

$$\Phi_M \equiv \frac{1}{4\pi} \frac{\vec{m} \circ \vec{x}}{r^3},$$
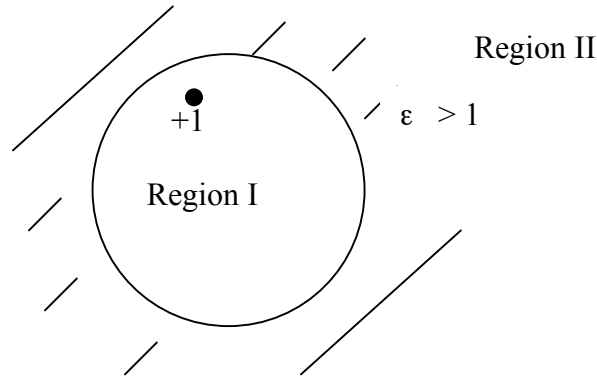
where

$$\vec{m} = -2\pi\vec{B} a^3.$$

Problem 3.

Prove that the electrostatic potential from a distant set of static charges, averaged on a spherical surface in charge free space, is the same as the potential evaluated at the center of the sphere.

Problem 4.

A positive unit charge is inside a spherical "bubble" of vacuum of radius a which is embedded in an infinite dielectric slab, as shown.

Assuming the potential is of the form,

$$G(\vec{x},\vec{x}') = 4\pi \sum_{\ell,m} Y^*_{\ell,m}(\theta',\phi')Y_{\ell,m}(\theta,\phi)g_\ell(r,r') \,*$$

one obtains the radial equation satisfied by $g_\ell(r,r')$:

$$\left[-\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + \ell(\ell+1)\right]g_\ell(r,r') = \delta(r-r').$$

For $r \ne r'$ the linearly independent solutions go like $g_\ell(r,r') \propto r^\ell$ or $r^{-(\ell+1)}$. Write the appropriate forms for $g_\ell(r,r')$ in the various regions and determine all the numerical coefficients for Regions I and II. (You do not have to actually solve for the numerical coefficients, just show the equations which determine them.)

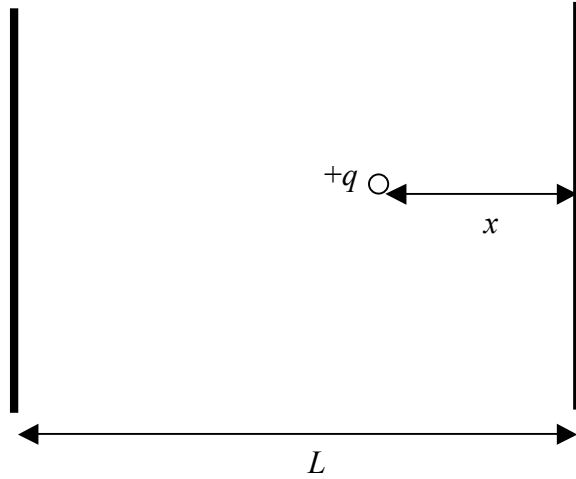[Hint: The Coulomb expansion in terms of spherical harmonics is

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell,m} \frac{4\pi}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}} Y^*_{\ell,m}(\theta',\phi')Y_{\ell,m}(\theta,\phi),$$

where $r_<$ is the lesser of r and r', and $r_>$ is the greater of the two.]

Problem 5.

A charge $+q$ is a distance x from one of two grounded, infinite conducting planes that are separated by a distance L. What is the exact force on the charge? Express your answer using summation notation.

Problem 6.

A metal bar of length 1.0 meters falls from rest under gravity while remaining horizontal with its ends pointing toward the magnetic east and west. What is the potential difference between its ends when it has fallen 10 meters? The horizontal component of the earth's magnetic field is 1.7 x 10^{-5} gauss = 1.7 μ Tesla.

Baylor University Physics Ph.D. Program
Preliminary Exam 2007

Part IV: Statistical Mechanics & Mathematical Methods
Saturday, May 26, 2007, 1:30 pm

Problem 1.

Consider a system of N non-interacting particles obeying Maxwell-Boltzmann statistics, each with two possible energies, 0 and ϵ. Apply the strategy of microcanonical ensemble to this system assuming n particles in the upper energy state and show that the internal energy as a function of temperature can be written as

$$U = \frac{N\varepsilon}{e^{\varepsilon/kT} + 1}$$

Find the Helmholtz free energy and C_V of this system.

Problem 2.

Consider an intrinsic semiconductor that has an energy gap of E_g. The densities of conduction and valence electrons are n and p, respectively.

Let the total number of electrons in the system be N. When the temperature T is 0 (zero), all the valence states (bands) are occupied and the conduction states (bands) are empty. That is,

$$N(T = 0) = \sum_j 1, \text{ where j is over all the occupied states.}$$

When T > 0,

$$N(T > 0) = \sum_i \frac{1}{e^{\beta(\varepsilon_i - \mu)} + 1} + \sum_j \frac{1}{e^{\beta(\varepsilon_j - \mu)} + 1},$$

where ε_i is an energy level in the conduction band and ε_j is an energy level in the valance band.

For low temperatures:

(a) Show that n and p are equal and are functions of the temperature T as follows:

$$n = p = 2e^{-\frac{E_g}{2kT}} \left\{ \frac{2\pi (m_e m_h)^{1/2} kT}{h^2} \right\}^{3/2}.$$

(b) Show that the chemical potential $\mu(T)$ is

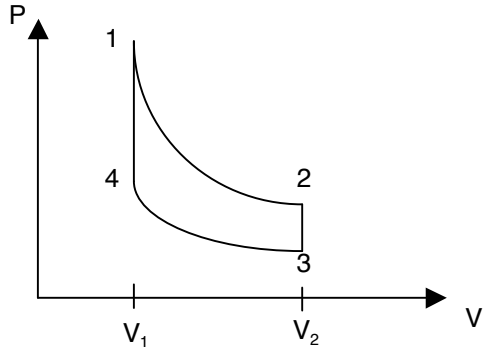$$\mu = \frac{1}{2}E_g + \frac{3}{4}kT \ln\left(\frac{m_h}{m_e}\right).$$

Note that $\Gamma(1/2) = \frac{1}{2} \Gamma(3/2) = \sqrt{\pi}$ and

$$\int_0^\infty x^m e^{-ax^2} dx = \frac{1}{2a^{(m+1)/2}} \Gamma[(m+1)/2].$$

Problem 3.

Consider the following thermodynamic cycle ($1 \rightarrow 2 \rightarrow 3 \rightarrow 4$). The transitions from 1 to 2 and from 3 to 4 are isothermal. It is known as the Sterling cycle.

(a) Calculate the work done by the ideal gas, the total heat input, and thus the efficiency of the engine.

(b) Compare its efficiency with that of the Carnot cycle. Which is less?

Problem 4.

Evaluate the following integral by contour integration:

$$\int_0^\infty \frac{x^{1/2}dx}{1+x^4}.$$

(a) Show your contour and identify all poles and branch cuts in the complex plane.

(b) Perform the integration and confirm that your answer is real.

Problem 5.

Consider the Sturm-Liouville equation

$$y'' + 2\,y' + y = f(x),$$

where the function $f(x)$ is piece-wise continuous on $[0,\pi]$ and the boundary conditions are $y(0) = 0$ and $y(\pi) = 0$. Determine the solution, respectively, by

(a) the method of variation; and

(b) the method of Green function.

Problem 6.

A horizontal load-bearing beam of length L is simply supported at both ends, and is subject to a variable load per unit length:

$$q(x) = (a/L)x \ (0 \le x \le L)$$

Let y(x) denote the downward deflection of the beam. The deflection y(x) satisfies the DE:

$$\frac{d^4 y}{dx^4} = \frac{1}{EI} q(x)$$

where 1/EI is the rigidity of the beam. The boundary conditions for the beam are:
$$y(0) = y(L) = y''(0) = y''(L) = 0$$

(a) Solve for the deflection of the beam using a Fourier sine series of the form:
$$y(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

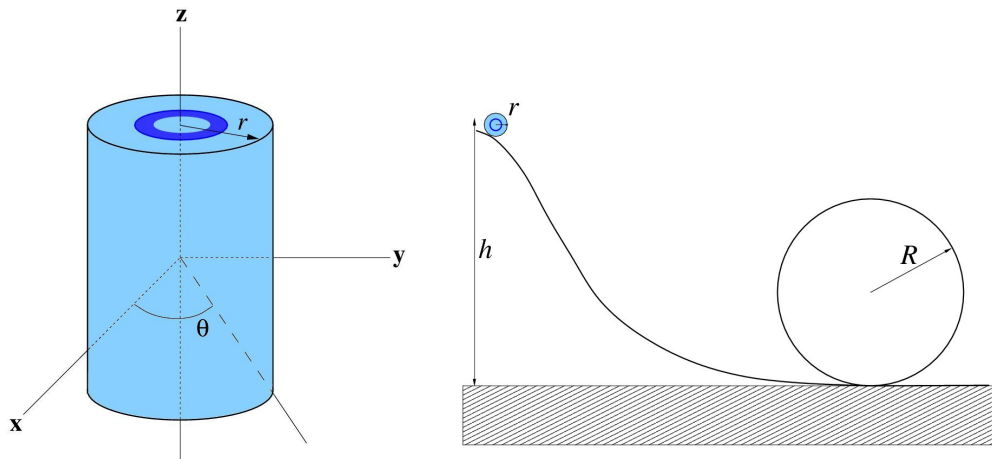(b) Solve the problem in closed form by integrating the differential equation.

Baylor University Physics Ph.D. Program
Preliminary Exam 2008

Part I: Classical Mechanics
Friday, May 30, 2008, 9:00 am

Problem 1

A cylindrically-shaped object of mass M and radius r is shown in the figure on the left. The cylinder is azimuthally symmetric about an axis passing through its center, but its density is non-uniform in the *radial* direction.

To measure the cylinder's moment of inertia I_{CM} about its axis of symmetry, you perform an experiment in which you roll the object down a curved incline that ends in a circular loop of radius R, as shown in the figure on the right.

The initial height of the incline is adjustable. You observe that the minimum height for which the cylinder goes around the loop is h. At no point during the motion of the cylinder does it slip on the curved incline. Assume $r \ll R$.

(a) Find the moment of inertia of the cylinder around its axis of symmetry in terms of the given quantities. Ignore any effects due to air resistance.

(b) What are the minimum and maximum values of h that are possible for a cylindrically-shaped object of mass M and radius r? Explain.

Problem 2

A particle with mass m interacts with a central force $F(r)$, making an orbit of the form

$$r(\theta) = \frac{r_0}{1 + b\cos(3\theta)}$$

about $r = 0$, with $|b| < 1$. The strength of the force at a distance r_0 is $F(r_0) = -f_0$, where f_0 is a positive constant.

(a) Show that

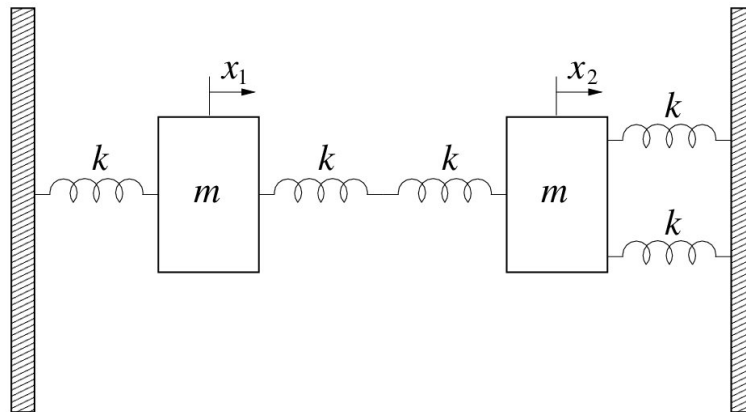$$F(r) = f_0\left[\left(\frac{c_1 r_0}{r}\right)^3 - \left(\frac{c_2 r_0}{r}\right)^2\right].$$

Determine numeric values for the constants c_1 and c_2.

(b) Find the total energy and angular momentum of the particle in terms of m, r_0, b, and f_0.

(c) Show that the particle has a stable circular orbit at $r = r_0$.

Five springs, all of constant k and negligible mass, are connected as shown between two masses m. The two springs on the right are connected in parallel, and the two springs in the middle are connected in series. The masses move in one dimension on a frictionless, horizontal surface. The displacements of the masses from their equilibrium positions are x_1 and x_2, respectively.

(a) Set up the kinetic and potential energy matrices and find the two eigenfrequencies.

(b) Find the normalized eigenvectors and give the physical nature of the two modes of vibration.

Problem 4

Consider an infinitely long continuous string with tension τ. A point mass M is located on the string at $x = 0$. A wave train with velocity ω/k is incident from the left.

(a) Show that reflection and transmission occur at $x = 0$ and that the coefficients R and T are given by

$$R = \sin^2 \theta \qquad \text{and} \qquad T = \cos^2 \theta$$

where θ is given by

$$\tan \theta = \frac{M\omega^2}{2k\tau}$$

(b) What are the phase changes for the reflected and transmitted waves?

Problem 5

A rigid body is comprised of 8 equal masses m at the corners of a cube of side a, held together by massless struts.

(a) Derive the inertia tensor $\mathbf{I}$ for rotation about a corner O of the cube and show that it is
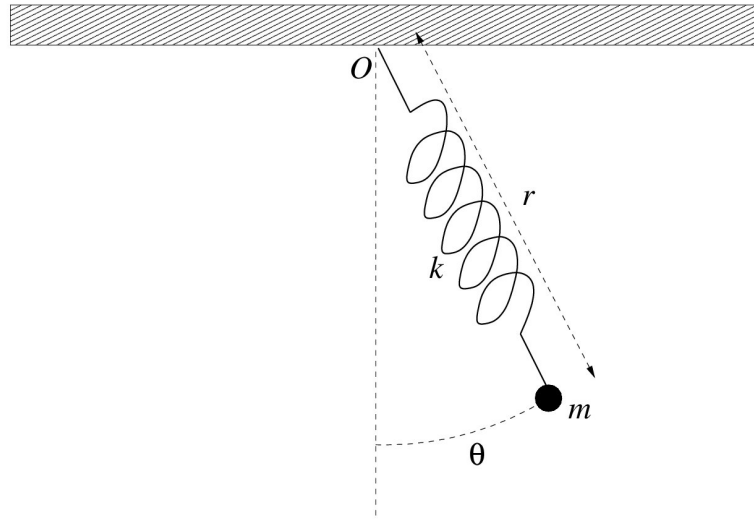
$$\mathbf{I} = ma^2 \begin{bmatrix} 8 & -2 & -2 \\ -2 & 8 & -2 \\ -2 & -2 & 8 \end{bmatrix}$$

(Use x, y, and z axes along the three edges of the cube through O.)

(b) Find the rotation matrix that rotates the axes to a set of coordinates where the x′ axis is along a body diagonal of the cube (the vector [1,1,1]) and the y′ axis is in the yz plane.

(c) Transform the inertia tensor $\mathbf{I}$ to the new, primed coordinate system.

5

A pendulum is made from a massless spring (force constant k and unstretched length b) that is suspended at one end from a fixed pivot O and has a mass m attached to its other end. The spring can stretch and compress but cannot bend, and the whole system is confined to a single vertical plane.

(a) Write down the Lagrangian for the pendulum, using the length of the spring r and the angle θ as generalized coordinates.

(b) Find the Lagrange equations of motion for the system.

(c) The equations of part (b) cannot be solved analytically in general; however, they can be solved for small oscillations. Do this and describe the motion.

(d) Under what conditions will this system reduce to (i) a simple pendulum, and (ii) a linear simple harmonic oscillator?

Problem 1

A particle of mass m moves in a potential well given by

$$V(x) = \begin{cases} 0, & |x| < a/2 \\ V_0, & |x| \geq a/2 \end{cases}$$

where V_0 and a are positive constants.

(a) Find the eigenvalue E_n and eigenfunction $\psi_n(x)$ of the n-th bound state for $E_n \ll V_0$.

(b) Show that the probability of finding the particle outside the potential well is approximately

$$P \simeq \frac{1}{\sqrt{2mV_0}} \frac{2\hbar E_n}{aV_0}.$$

(c) Calculate the expectation values of $V(x)$ and $V^2(x)$.

Problem 2

Consider an alpha particle in the potential

$$V(r) = \begin{cases} 0, & 0 \le r < R \\ zZe^2/(4\pi\epsilon_0 r), & R \le r \end{cases}$$

where $z = 2$ for a helium nucleus, $Z = 92$ for a uranium nucleus, and we consider the orbital angular momentum $L = 0$ state only. Here, we set $R \simeq 6.8$ fm, and we assume the alpha particle is initially inside the nuclear region $r \le R$. Use $E = 4.40$ MeV for the decay energy of the alpha particle.

(a) Sketch the potential $V(r)$ versus r. Identify the barrier and explain how the WKB approximation applies in this case.

(b) Using the WKB approximation, show that the formula for the tunneling probability is $P_t = e^{-2I}$ with I given by

$$I = \frac{zZe^2}{4\pi\epsilon_0}\sqrt{\frac{2\mu}{\hbar^2 E}}\left[\cos^{-1}\sqrt{x} - \sqrt{x - x^2}\right], \quad x = E/V(R),$$

where μ is the reduced mass as usual.

(c) Use an estimate for the barrier striking frequency ω to obtain the alpha decay rate $1/\tau$ as

$$1/\tau = \omega P_t$$

and find a value for $1/\tau$ in s^{-1}.

Use $m_N(^4\text{He}) = 3.73$ GeV/c^2 and $m_N(^{235}\text{U}) = 218.94$ GeV/c^2.

Useful quantities: 1 GeV $= 10^3$ MeV, $\hbar c \approx 197.3$ MeV·fm, $e^2/(4\pi\epsilon_0\hbar c) \approx 1/137$
$\hbar = 1.055 \times 10^{-34}$ J·s $= 6.582 \times 10^{-22}$ MeV·s

Problem 3

A system described by the Hamiltonian

$$\mathbf{H} = -\frac{\hbar^2}{2m}\nabla^2 + \frac{m}{2}(\omega_1\, x^2 + \omega_2\, y^2 + \omega_3\, z^2)$$

is a 3-dimensional anisotropic harmonic oscillator.

(a) Determine the possible energy levels E_n of this system. In what cases (generic classes) of $\omega_{1,2,3}$ values will there *never* be degeneracy of any energy level E_n?

(b) For the isotropic case ($\omega_1 = \omega_2 = \omega_3 = \omega$), calculate the degeneracy of the level E_n.

Problem 4

Consider a particle that is trapped between two hard walls in one dimension:

$$V(x) = \begin{cases} 0, & |x| < a \\ \infty, & |x| \geq a. \end{cases}$$

Assume that the particle is exactly at $x = 0$ at $t = 0$ with certainty.
That is, $\psi(x, t = 0) = \delta(x)$, where $\delta(x)$ denotes the Dirac delta function.

(a) What are the relative probabilities for the particle to be found in various energy eigenstates?

(b) Write down the wavefunction for $t \geq 0$.

(c) Assume that a perturbation given by

$$\mathbf{H}' = \alpha \, \delta \left(x + \frac{a}{2} \right) - \beta \, \delta \left(x - \frac{a}{2} \right)$$

is imposed, where $0 < \alpha \ll 1$ and $0 < \beta \ll 1$.

Find the first-order correction to the allowed energies. What happens when $\alpha = \beta$? Why?

Problem 5

The canonical commutation relations for position and momentum are given by

$$[x_i, p_j] = i\,\hbar\,\delta_{ij}, \qquad [x_i, x_j] = [p_i, p_j] = 0 \qquad \text{where} \quad i, j = x, y, z.$$

These commutation relations can be used to obtain the following:

$$[L_i, x_j] = i\,\hbar\,\epsilon_{ijk}\,x_k, \qquad [L_i, p_j] = i\,\hbar\,\epsilon_{ijk}\,p_k, \qquad [L_i, L_j] = i\,\hbar\,\epsilon_{ijk}\,L_k$$

where $\vec{L} \equiv \vec{r} \times \vec{p}$ denotes the angular momentum.

(a) The quadrupole tensor $\mathbf{Q}$ is defined as

$$Q_{ij} = 3x_i x_j - r^2 \delta_{ij}.$$

Give the physical meaning of Q_{ij}, and work out explicitly the commutators

$$[L^{\pm}, Q_{11}] \qquad \text{and} \qquad [L^{\pm}, Q_{13}]$$

where $L^{\pm} = L_x \pm iL_y$.

(b) For a given system that consists of N distinguishable particles, each of which has mass m_i and coordinates $\mathbf{r}_i = (x_i, y_i, z_i)$ with $i = 1, 2, \ldots N$, find the commutation relations

$$[L_x, Y] \qquad \text{and} \qquad [L_x, L_y]$$

where $\vec{L}$ denotes the angular momentum of the whole system, and X, Y, and Z are the coordinates of its center of mass, defined, respectively, by

$$\vec{L} = \sum_{i=1}^{N} \vec{L}_i \qquad \text{and} \qquad Y = \frac{1}{M} \sum_{i=1}^{N} m_i\, y_i.$$

M is the total mass of the system.

The Hamiltonian for a two-electron atom, with an infinitely massive nucleus of charge Z, may be expressed in simplifying units as

$$\mathbf{H} = -\frac{1}{2}\nabla_1^2 - \frac{1}{2}\nabla_2^2 - \frac{Z}{r_1} - \frac{Z}{r_2} + \frac{1}{r_{12}}$$

where the two electrons have position vectors $\vec{r}_1$ and $\vec{r}_2$ and $r_{12} = |\vec{r}_1 - \vec{r}_2|$. An approximate ground-state wavefunction $\psi(\vec{r}_1, \vec{r}_2)$ may be constructed by taking the product of two normalized hydrogen-like wave functions $\phi(r_1)$ and $\phi(r_2)$, where

$$\phi(r) = \sqrt{\frac{Z_e^3}{\pi}} \exp(-Z_e r)$$

is the ground-state wavefunction for a single-electron atom with an effective nuclear charge $Z_e < Z$ to take into account screening by the other electron.

(a) Taking Z_e as a free parameter, show that the total ground state energy is

$$E_0(Z_e) = Z_e^2 - 2ZZ_e + \frac{5}{8}Z_e.$$

(b) By minimizing the ground state energy, find the value of the effective charge Z_e (in terms of Z) that gives the best approximation to the true ground state.

(c) Find the energy of the approximate ground state from part (b).

Useful relations:

$$\nabla^2 f(r) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right), \qquad d^3\mathbf{r} = r^2\,dr\,d\cos\theta\,d\phi$$

$$\int x^n e^{-ax}dx = -\frac{n!\,e^{-ax}}{a^{n+1}}\sum_{l=0}^{n}\frac{1}{l!}(ax)^l, \qquad \int_0^\infty x^n e^{-ax}dx = \frac{n!}{a^{n+1}}$$
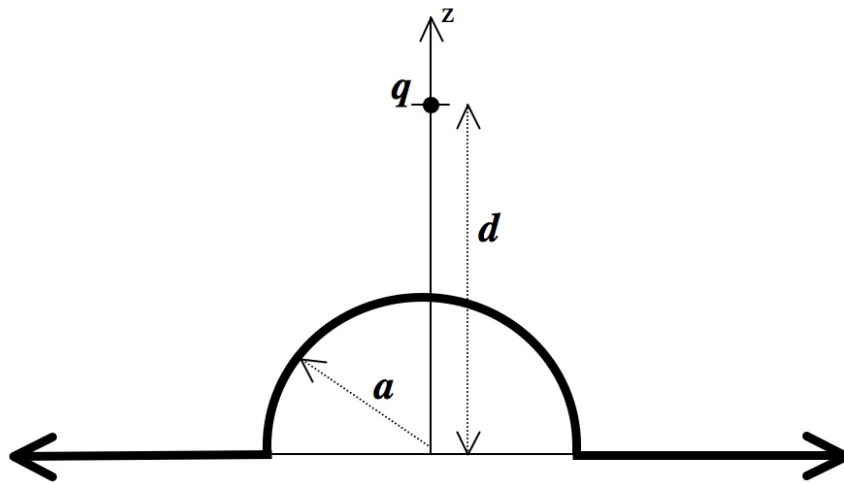
$$\int_{-1}^{1}\frac{dx}{\sqrt{a^2+b^2-2abx}} = \frac{2}{\max(a,b)} \quad \text{if } a,b > 0$$

6

Problem 1

A conducting surface consists of an infinite plane with a hemispherical bump of radius a that is centered at the origin. A point charge q is placed on the positive z axis a distance d from the origin ($d > a$). What is the force on the charge?

Consider a two-dimensional charge-free region S consisting of a circle of radius b centered about the origin O and the region inside that circle: $S = \{(\rho, \phi), \ 0 \le \rho \le b, \ 0 \le \phi \le 2\pi\}$.

If the electrostatic potential $\Phi(\rho, \phi)$ is specified as $V(b, \phi)$ on this circle of radius b, the boundary of S, show that it is given in the interior of S by
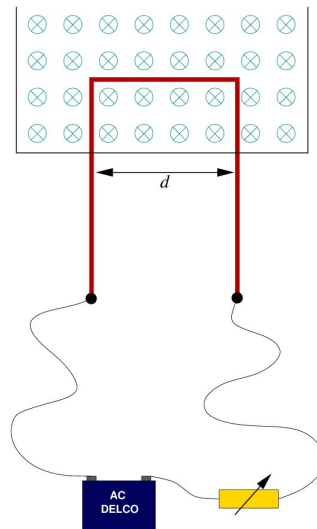
$$\Phi(\rho, \phi) = \frac{1}{2\pi} \int_0^{2\pi} d\phi' \frac{(b^2 - \rho^2)V(b, \phi')}{b^2 + \rho^2 - 2b\rho \cos(\phi - \phi')}$$

Problem 3

(a) From Maxwell's equations, derive the wave equation for the electric field $\vec{E}$ in free space. State what system of units you used.

(b) From the result in part (a), what is the speed of propagation of an electromagnetic wave?

(c) For an isotropic, non-conducting medium, express Maxwell's equations in terms of dot products and cross products involving the wave vector.

A U-shaped wire with mass m = 14 g hangs vertically with one end in a 1.0 T uniform magnetic field $\vec{B}$, which points into the page as shown in the figure. The two ends of the U-shaped wire are connected in series to a 12 V car battery and a variable resistor. Assume that the resistance of the U-shaped wire and other wires is very small compared to the variable resistor. The distance d is 6.0 cm.

(a) For what setting of the variable resistor R_1 would the magnetic force upward exactly balance the gravitational force downward? (Indicate whether the current is clockwise or counterclockwise.)

For parts (b)–(e), the variable resistor is set to a new value $R_2 = 0.80 \cdot R_1$. The new current in the loop is held *constant* at I_2. You observe that the U-shaped wire moves.

(b) In which direction does the U-shaped wire move? Explain.

(c) What is the acceleration of the U-shaped wire?

(d) What is the work done on the U-shaped wire (excluding work done by the gravitational force) to move it a distance of 2.0 cm?

(e) What is the work done by the magnetic field on the U-shaped wire to move it a distance of 2.0 cm? Is this answer consistent with your answer to part (d)? Why or why not?

(a) Derive the following expression for the electric field at a position $\vec{r}$ resulting from an electric dipole $\vec{p}$:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{(3\vec{p}\cdot\hat{r})\hat{r} - r^2\vec{p}}{r^5}\right]$$

Assume the magnitude of $\vec{r}$ is much larger than the charge separation.

(b) Derive the torque exerted on a dipole $\vec{p}$ placed in an electric field $\vec{E}$.

(c) Evaluate the torque exerted by one dipole on an identical dipole. Again, assume the separation of the dipoles is much larger than the dipoles themselves.

Problem 6

The Green function for a perfectly conducting sphere of radius a centered at the origin of a spherical coordinate system is

$$G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} - \frac{a}{r'|\vec{r} - \vec{r}''|} \qquad \text{where} \qquad \vec{r}'' = \vec{r}'\left(\frac{a}{r'}\right)^2.$$

The expansion for $1/|\vec{r} - \vec{r}'|$ is given by

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{l,m} \frac{4\pi}{2l + 1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta', \phi')Y_{lm}(\theta, \phi)$$

where the $Y_{lm}(\theta, \phi)$ are spherical harmonics, $r_<$ is the *lesser* of r and r', and $r_>$ is the *greater* of r and r'.

(a) Show that the Green function may be written as

$$G(\vec{r}, \vec{r}') = \sum_{l,m} \frac{4\pi}{2l + 1} \left[\frac{r_<^l}{r_>^{l+1}} - \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1}\right] Y_{lm}^*(\theta', \phi')Y_{lm}(\theta, \phi).$$

(b) The surface of a sphere has a given voltage

$$V(\theta) = V_0 \cos\theta = V_0\sqrt{\frac{4\pi}{3}}Y_{10}(\theta).$$

Given the expression

$$\Phi(\vec{r}) = -\frac{1}{4\pi} \oint da' \, V(\theta) \frac{\partial}{\partial n'}G(\vec{r}, \vec{r}'),$$

where $\frac{\partial}{\partial n'}$ denotes a primed normal gradient on the surface, use the Green function from part (a) to show that the potential outside the sphere is given by a dipole form

$$\Phi(\vec{r}) = \frac{\vec{p} \cdot \vec{r}}{r^3}.$$

Find the value of $\vec{p}$ for the sphere.

Problem 1

Consider the modified Bessel function

$$I_\nu(z) = \frac{1}{2\pi i} \int_C \exp\left[\frac{z}{2}\left(t + \frac{1}{t}\right)\right] t^{-\nu-1} \, dt$$

where the contour wraps around the origin in a counterclockwise direction.

Along the real axis, show that the asymptotic behavior of $I_\nu(x)$ as x becomes large and ν remains fixed is

$$I_\nu(x) \approx \frac{1}{\sqrt{2\pi x}} \, e^x.$$

Problem 2

The matrix

$$
\rho_2 \equiv \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}
$$

is one of the Dirac matrices that appears in Quantum Electrodynamics (QED).

(a) Show that ρ_2 is unitary.

(b) Find all of the eigenvalues and eigenvectors of ρ_2.

(c) Identify two other Dirac matrices ρ_1 and ρ_3 such that the following anticommutation identities are satisfied:

$$\rho_1\rho_2 + \rho_2\rho_1 = 0 \quad \text{and} \quad \rho_3\rho_2 + \rho_2\rho_3 = 0 \quad \text{and} \quad \rho_1\rho_3 + \rho_3\rho_1 = 0$$

(Hint: $[\rho_1]_{11} = 0 \quad \text{and} \quad [\rho_3]_{11} = 1$.)

(d) For ρ_1, ρ_2, and ρ_3, prove that $\text{Tr}[\rho_i\rho_j] = 4\delta_{ij}$.

Problem 3

A rectangular membrane $0 \leq x \leq a, 0 \leq y \leq b$ is clamped on all sides and is loaded by a uniformly distributed external force q (per unit area). The deflection $u(x, y, t)$ satisfies the DE:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \frac{-q}{T}$$

where $c^2 = \mu/T$. T is the tension in the membrane and μ is the mass per unit area. Assume static (time-independent) deflection.

(a) Show that the auxiliary function

$$u_1(x, y) = \frac{qx(a - x)}{2T}$$

satisfies the PDE, but not all of the boundary conditions.

(b) If the overall solution is sought in the form $u_1(x, y) + u_2(x, y)$, determine the PDE and boundary conditions that should be satisfied by u_2.

(c) Find $u(x, y)$. Simplify your solution as much as possible.

Problem 4

The energy of 1 mole of a gas in a particular reversible system is given by $U = AP^2V$ where A is a constant and has units of P^{-1}. Find an equation for the adiabatic lines and sketch a few of these lines in the PV plane.

Problem 5

A system of volume V contains a variable number of non-conserved particles of mass m. A particle may be created by an expenditure of energy equal to its rest energy (mc^2) plus its kinetic energy.

(a) Show that the grand partition function Z for this system can be written as

$$Z = \exp\left[V\left(\frac{2\pi mkT}{h^2}\right)^{3/2} \exp\left(\frac{-mc^2}{kT}\right)\right].$$

(b) Find the internal energy U of this system and show that it is related to the pressure P and volume as

$$U = \left(\frac{3}{2} + \frac{mc^2}{kT}\right)PV.$$

Problem 6

(a) Derive an analytical expression for the fraction of electrons excited above the Fermi level, E_F, at temperature T_0. Leave your answer in terms of E_F and T_0.

(b) For a Fermi energy of 4.00 eV at room temperature, calculate the fraction of electrons excited above the Fermi level. ($k_B = 8.62 \times 10^{-5}$ eV/K)

(c) To what temperature must this electron gas be raised in order for 2.0% of the electrons to be excited to energy levels above E_F?

Baylor University Physics Ph.D. Program
Preliminary Exam 2009

Part I: Classical Mechanics
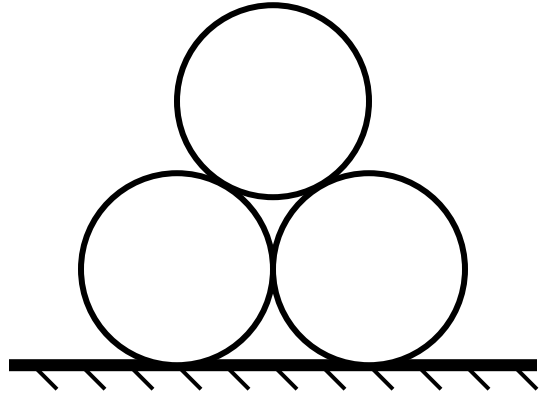### Friday, May 29, 2009, 9 am–12 pm

Problem 1

A particle of mass m is constrained to move on the surface of a cone of revolution $z = r\cos\alpha$, where α is a constant and r is in spherical polar coordinates. It is acted upon by a constant gravitational field given by $\vec{g} = -g\hat{z}$ along the axis of the cone.

(a) Find the Lagrangian of the system and the Lagrangian equations of motion.

(b) Reduce the Lagrangian equations of motion to quadratures and determine the two constants of the motion.

(c) Find the angular frequency of a circular orbit at $r_0 = z_0/\cos\alpha$.

(d) What is the frequency of small radial oscillations?

(e) Is it possible for any angle α to have closed orbits?

Problem 2

Three identical cylinders, each with mass M, radius R, and moment of inertia kMR^2, are placed on the floor to form a vertical triangular structure as shown in the figure.
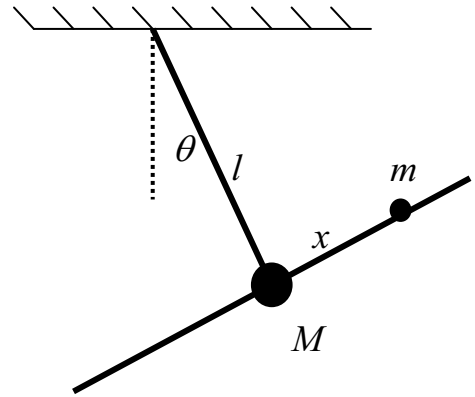

(a) If there is no friction between the cylinders or between the floor and cylinders, what is the initial downward acceleration of the top cylinder?

(b) If there is friction between the bottom two cylinders and the floor, but there is no friction between any of the cylinders, show that the initial downward acceleration of the top cylinder is $g/(3+2k)$.

(c) If there is no friction between the bottom two cylinders and the floor, but there is friction between the cylinders, find the initial downward acceleration of the top cylinder.

Problem 3

A mass M is fixed at the right-angled vertex where a massless rod of length l is attached to a very long massless rod. A mass m is free to move frictionlessly along the long rod (assume that it can pass through M). The rod of length l is hinged at a support, and the whole system is free to rotate, in the plane of the rods, about the hinge. Let θ be the angle of rotation of the system, and x be the distance between m and M.

(a) Find the equations of motion and show that those could be written as

$$l\ddot{\theta} + \ddot{x} + g\theta = 0$$
$$-Ml\ddot{x} + mgx = 0$$

for small θ and x.

(b) Find the characteristic frequencies and normal modes of oscillation for small θ and x.

Problem 4

(a) The motion of a one-dimensional damped harmonic oscillator is described by the differential equation

$$\ddot{x}(t) + \beta\dot{x}(t) + \omega^2 x(t) = 0,$$

where $x(t)$ gives the instantaneous position of the particle. Given that $\omega^2 > \beta^2$ (corresponding to the "under damped" case), find the general solution of the oscillator for initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$.

(b) For $v_0 = 0,$ show that $\dot{x}(t) = 0$ when t is given by

$$t = \frac{n\pi}{\omega_1}, \quad n = 0,1,2,3,\ldots$$

Problem 5

Consider a particle of mass *m* moving along a straight line under the influence of the potential,

$$V(x) = (x^2 - 3)e^x.$$

(a) Plot the potential vs. position and identify the 6 different energy values and ranges of values for distinct motion in the system. <u>Briefly</u> indicate the motion associated with each energy value (or range of values).

(b) Find the amplitude and frequency of the oscillation of the particle near the point $x = 1$.

(c) Imposing a perturbation of

$$\Delta V(x) = \frac{1}{2}\delta(x-1)^2,$$

where $0 < \delta \ll 1$, find the exact expressions of the amplitude and frequency of the modified oscillation of the particle near the point $x = 1$.

Problem 6

(a) Given Hamilton's equations in generalized coordinates ($H = H(p_i, q_i, t)$),

$$\frac{\partial H}{\partial p_i} = \dot{q}_i, \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i,$$

show that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}.$$

(b) Show that the integral statement ($L = L(\dot{q}_i, q_i, t)$),

$$\int dt \sum_i \dot{q}_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) \right) = 0,$$

which follows from the Euler-Lagrange equations in generalized coordinates, $\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0$, and where the time integral is over an arbitrary interval, implies that

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t},$$

where

$$H = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L.$$

Baylor University Physics Ph.D. Program
Preliminary Exam 2009

Part II: Quantum Mechanics
Friday, May 29, 2009, 1:30 pm–4:30 pm

Problem 1

A particle is in a harmonic oscillator potential. The initial state of the particle is a linear combination of $|0\rangle$ and $|1\rangle$, where the particle is three times as likely to be in state $|0\rangle$.

(a) Calculate the properly normalized initial state $|\psi(0)\rangle$.

(b) Using creation and annihilation operators, calculate $\langle X \rangle$ and $\langle P \rangle$.

(c) Again using creation and annihilation operators, calculate $\langle X^2 \rangle$ and $\langle P^2 \rangle$.

(d) Calculate ΔX and ΔP, and verify that the Heisenberg Uncertainty principle is satisfied for this system.

Problem 2

In the one-dimensional case, consider a particle that is moving under the potential,

$$V(x) = \begin{cases} -V_0 \delta(x), & |x| < a \\ \infty, & |x| \geq a \end{cases}$$

where V_0 is a real and positive constant, and $\delta(x)$ denotes the Dirac delta function.

(a) Find the odd wave function, $\psi_n(-x) = -\psi_n(x)$, and its energy spectrum.

(b) Find the even wave function, $\psi_n(-x) = \psi_n(x)$, and its energy spectrum in the two limits $V_0 \gg 1$ and $V_0 \ll 1$.

Problem 3

Consider a quantum system with three linearly independent states. The Hamiltonian, in matrix form, is given by:

$$H = V_0 \begin{pmatrix} 3 & 0 & \varepsilon \\ 0 & 2 & 0 \\ \varepsilon & 0 & 3 \end{pmatrix}$$

(a) Write down the eigenvectors and eigenvalues of the *unperturbed* Hamiltonian ($\varepsilon = 0$).

(b) Solve for the *exact* eigenvalues of H.

(c) Using the appropriate perturbation theory, find the first-order corrections to the unperturbed eigenvalues. Compare the corrected eigenvalues with the exact results from part (b).

Problem 4

Consider a physical system whose three-dimensional state space is spanned by the orthonormal basis formed by the three kets $|u_1\rangle$, $|u_2\rangle$, and $|u_3\rangle$. Consider the operators L_z and S defined as:

$$L_z|u_1\rangle = |u_1\rangle \qquad L_z|u_2\rangle = 0 \qquad L_z|u_3\rangle = -|u_3\rangle$$
$$S|u_1\rangle = |u_3\rangle \qquad S|u_2\rangle = |u_2\rangle \qquad S|u_3\rangle = |u_1\rangle$$

(a) Write the matrices which represent, in the $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ basis, the operators L_z, L_z^2, S, and S^2. Are these operators observables?

(b) Find the form of the most general matrix which represents an operator which commutes with L_z. Repeat this for L_z^2 and S^2.

(c) Do L_z^2 and S form a complete set of commuting observables? Find a basis of common eigenvectors.

4

Problem 5

(a) Prove the following corollary to the variational principle:

If $\langle \psi | \psi_{gs} \rangle = 0$, then $\langle H \rangle \geq E_1$, where E_1 is the energy of the first excited state.

(b) Using the trial function:

$$\psi(x) = \begin{cases} A\cos(\pi x/a), & -a/2 < x < a/2 \\ 0, & \text{otherwise} \end{cases}$$

obtain the best approximation to the ground state energy of the one-dimensional harmonic oscillator. Simplify your answer as much as possible and compare with the exact ground state energy.

(c) Using the trial function:

$$\psi(x) = \begin{cases} B\sin(\pi x/a), & -a < x < a \\ 0, & \text{otherwise} \end{cases}$$

obtain the best approximation to the energy of the first excited state. Simplify your answer as much as possible and compare with the exact answer.

Problem 6

An electron is subject to a uniform, time-independent magnetic field of strength B in the positive z-direction.

(a) Find the eigenstate of $\vec{S} \cdot \hat{n}$ with eigenvalue $\hbar/2$, where $\hat{n}$ is a unit vector, lying in the xz-plane, that makes an angle θ with the z-axis.

(b) If at $t = 0$ the electron is in the eigenstate found in part (a), find the probability for the electron being in the $s_x = -\hbar/2$ state as a function of time.

(c) Find the expectation value of S_x as a function of time.

Note that $S_k = (\hbar/2)\sigma_k$, where the Pauli matrices are given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
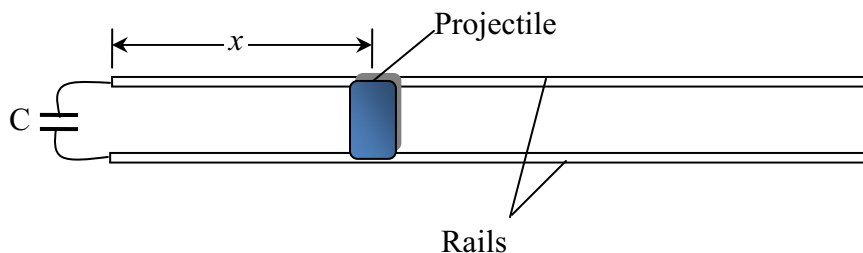
Baylor University Physics Ph.D. Program
Preliminary Exam 2009

Part III: Electricity and Magnetism
Saturday, May 30, 2009, 9 am–12 pm

Problem 1

An electromagnetic launcher, or rail gun, consists of two parallel conducting rails connected to a source of high voltage (usually a large capacitor) at one end. The projectile, a conducting mass, slides along the rails and completes the circuit (see figure below). The large current in this circuit generates a strong magnetic field that interacts with the current flowing through the projectile and accelerates it along the rails.

(a) Show that if the voltage source maintains a constant current in the rail circuit, the force on the projectile is directly proportional to the product of the square of the current and the self-inductance per unit length of the rails. [Hint: Neglect the resistance and show that the energy balance for the system can be written as $dW + \frac{1}{2}I^2 dL = I^2 dL$, where dW is the work done by the magnetic force on the projectile, $\frac{1}{2}I^2 dL$ is the increase of stored magnetic energy, and $I^2 dL$ is the work done by the voltage source to maintain the constant current.]

(b) Suppose that the inductance per unit length is $2/c^2$ (2×10^{-7} H/m). Estimate the current required to give a projectile of 200 g a muzzle speed of 3000 m/s in a gun of length 10 m.

Problem 2

Consider a simple capacitor formed from two insulated conductors. When equal and opposite charges are placed on the conductors, there is a potential difference between the conductors. The capacitance is defined as the ratio of the magnitude of the charge on one conductor to the potential difference, $C \equiv \left| \dfrac{Q}{\Delta V} \right|$.

(a) Using Gauss' law,

$$\oint_S \vec{E} \cdot d\vec{A} = 4\pi Q_{enclosed} \, ,$$

calculate the capacitance of two concentric cylinders of length, L, large compared to their radii a, b ($b > a$).

(b) Now consider the configuration in which there are two solid, conducting cylinders of radii a and b, but the cylinders are no longer concentric. The axes of the two long, cylindrical conductors are parallel and are placed a distance d apart, where $d \gg a$. Charges Q and $-Q$ are placed on the cylinders. Show that the approximate capacitance per unit length of the system is given by

$$C = \frac{1}{4\ln(\dfrac{d}{\sqrt{ab}})} \quad \text{(in Gaussian units)}$$

$$\left(\text{or } C = \frac{\pi\varepsilon_0}{\ln(\dfrac{d}{\sqrt{ab}})} \quad \text{(in MKS units)} \right).$$

Problem 3

The Rodrigues formula for Legendre polynomials is given by:

$$P_l(x) = \frac{1}{2^l \, l!} \left(\frac{d}{dx} \right)^l (x^2 - 1)^l.$$

(a) Expand $\cos 3\theta$ as a linear combination of the first four Legendre polynomials $P_l(\cos\theta)$.

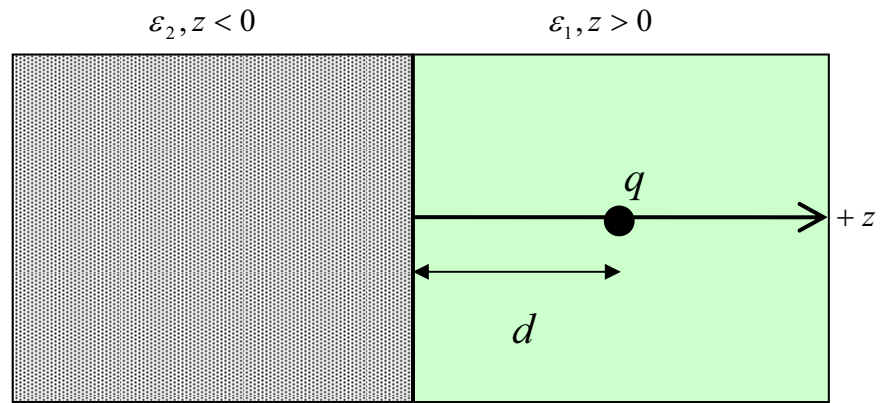(b) The potential at the surface of a sphere of radius, R, is given by

$$V_0(\theta) = k\cos 3\theta,$$

where k is a constant. Both the inside and outside of the sphere are free of charge. Find the potential inside and outside the sphere.

Hint: For azimuthal symmetry the angular part of Laplace's equation is solved by Legendre polynomials $P_l(\cos\theta)$.

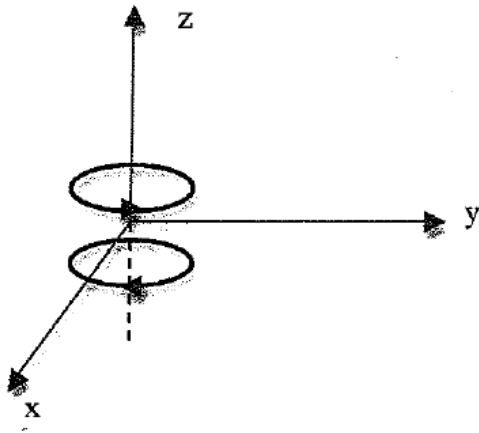(c) Find the surface charge density $\sigma(\theta)$ on the sphere.

Problem 4

A plane interface exists between dielectric media with dielectric constants ε_1 and ε_2. A point charge, q, is at a distance $d > 0$ from the interface, as shown above. q' and q'' are image charges located at $z = -d$ for q' ($z > 0$ region) and $z = d$ for q'' ($z < 0$ region). Use the boundary conditions on the interface to find the values of q' and q'' in terms of q, ε_1, and ε_2.

Problem 5

Consider two circular current loops, each of radius, a, and parallel to the x-y plane. Their centers are at $(0, 0, \pm z_0/2)$, and they carry equal but oppositely directed currents of magnitude, I, as shown below. For $r \gg a \gg z_0$, determine the vector potential and the magnetic field to leading order in $1/r$.

Problem 6

An iron sphere of radius R carries a charge Q and a uniform magnetization $\vec{M} = M\hat{z}$ that produces the magnetic field B,

$$B = \begin{cases} \dfrac{\mu_o}{4\pi} \dfrac{m}{r^3}\left[2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right] & \text{for } r > R \\[2mm] \dfrac{2}{3}\mu_o M\hat{z} & \text{for } r < R \end{cases}$$

where $m = \dfrac{4}{3}\pi R^3 M$.

(a) Find the angular momentum density in the electromagnetic field and show that the angular momentum stored in the field is $\dfrac{2}{9}\mu_o MQR^2\hat{z}$.

(b) Suppose the sphere is gradually (and uniformly) demagnetized (perhaps by heating it up past the Curie point). Use Faraday's law to determine the induced electric field, find the torque this field exerts on the sphere, and calculate the total angular momentum imparted to the sphere in the course of the demagnetization.

Baylor University Physics Ph.D. Program
Preliminary Exam 2009

Part IV: Statistical Mechanics & Mathematical Methods in Physics
Saturday, May 30, 2009, 1:30 pm–4:30 pm

<u>Problem 1</u>

Consider non-interacting, non-relativistic electrons ($\varepsilon = p^2/2m$) confined in two dimensions (2D).

  (a) Calculate the chemical potential (that is the Fermi energy) at 0 K.

  (b) Show that the average energy for an electron, E/N, is only half the Fermi energy, compared to $3/5\, \varepsilon_F$ in three dimensions (3D).

Problem 2

Consider an extremely relativistic gas of non-interacting, indistinguishable N monoatomic molecules with energy-momentum relationship $\varepsilon = pc$ (c is the speed of light).

(a) Calculate the Helmholtz free energy by evaluating the partition function.

(b) Show that this system also obeys $PV = nU$, where U is the internal energy, and determine n.

(c) What if they are now fermions (still extremely relativistic, e.g., electrons in a white dwarf star)? Explicitly show that they do (or do not) obey the same relationship, $PV = nU$.

Problem 3

An electrolytic cell is used as the working substance of a Carnot cycle. In the appropriate temperature range the equation of state for the cell is

$$\varepsilon = \varepsilon_0 - \alpha(T - T_0),$$

where $\alpha > 0$ and $T > T_0$. The energy equation is

$$U - U_0 = \left(\varepsilon - T\frac{d\varepsilon}{dT}\right)Z + C_Z(T - T_0),$$

where Z is the charge which flows through the cell and C_Z (which is assumed to be a constant) is the heat capacity at constant Z.

(a) Show that for an adiabatic process, ε can be expressed as

$$\varepsilon = \varepsilon_0 - \alpha T_0\left(Ae^{-\alpha Z/C_Z} - 1\right). \quad \text{[Here } A \text{ is a constant.]}$$

(b) Sketch the Carnot cycle on an $\varepsilon - Z$ diagram and indicate the direction in which the cycle operates as an engine.

(c) Use the expression for the efficiency of a Carnot cycle to show that charge transferred in the isothermal process must have the same magnitude.

3

Problem 4

Consider a physical system whose three-dimensional state space is spanned by the orthonormal basis formed by the three kets $|u_1\rangle$, $|u_2\rangle$, and $|u_3\rangle$. In the basis of these three vectors, taken in this order, the two operators H and B are defined by:

$$H = \hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

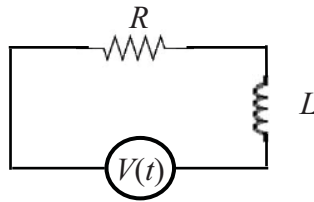$$B = b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

where ω_0 and b are real constants.

(a) Are H and B Hermitian?

(b) Show that H and B commute. Find a basis of eigenvectors common to H and B.

(c) Of the sets of operators: $\{H\}$, $\{B\}$, $\{H, B\}$, $\{H^2, B\}$, which form a complete set of commuting observables (CSCO)?

Problem 5

Solve the vibrating string problem $y_{tt} = a^2 y_{xx}$ if the initial shape is given by $y(x,0) = \dfrac{1}{\pi} x(\pi - x)$, $y_t(x,0) = 3$, and the boundary conditions are given by $y(0,t) = y(\pi,t) = 3t$.

Problem 6

Consider the following *RL* circuit:

(a) Write down the differential equation for the current $I(t)$ for an arbitrary applied voltage $V(t)$.

(b) Using the Laplace transform method, find $i(s)$, the Laplace transform of the current, in terms of the transform of the applied voltage $v(s)$, where

$$v(s) = \int_0^\infty e^{-st} V(t) dt$$

(c) Compute $v(s)$ for the case

$$V(t) = \begin{cases} V_0 \sin \omega t & t > 0 \\ 0 & t < 0 \end{cases}$$

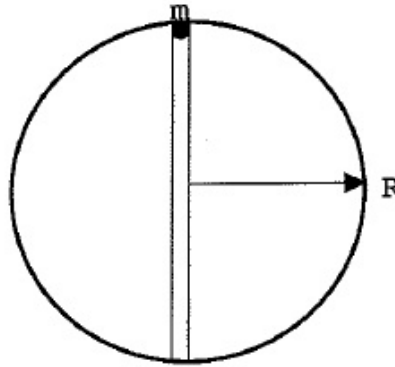(d) For the applied voltage in part (c), compute $I(t)$ by taking the inverse Laplace transform of $i(s)$.
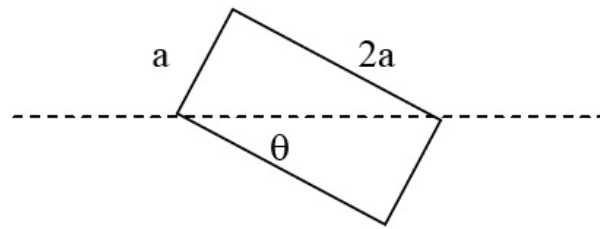
Problem 1

(a) A spherically symmetric planet of mass M and radius R has a homogeneous mass density. A straight tunnel is dug through its center and a small mass, m, is dropped in.

Find the period of motion. (The planet has no atmosphere.)

(b) The mass m is in a circular orbit just above the planet's surface. Show that the period of orbit is the same as the period of oscillation in part (a).

$$\tan \theta = \tfrac{1}{2}$$

The rectangle shown above is constrained to rotate about an axis through the diagonal as shown. Find the torque (magnitude) on the rectangle due to the clamps at the corners of the rectangle.

Problem 3

A particle is free to move on the surface of a torus given by

$$x(\theta, \phi) = (a + b\cos\phi)\cos\theta$$
$$y(\theta, \phi) = (a + b\cos\phi)\sin\theta$$
$$z(\theta, \phi) = b\sin\phi$$

(a) Find a suitable Lagrangian for this problem.

(b) Find a suitable Hamiltonian for this problem.

(c) Find two first integrals (constants) of the motion.

The generalized coordinates of a simple pendulum are the angular displacement θ and the angular momentum $ml^2\dot{\theta}$. Study, both mathematically and graphically, the nature of the corresponding trajectories in the phase space of the system, and show that the area A enclosed by a trajectory is equal to the product of the total energy E and the time period τ of the pendulum.

Consider a frictionless rigid horizontal hoop of radius R. Onto this hoop, three beads with masses $2m$, m, and m are threaded. The beads are connected with three identical springs, each with force constant k. Find the normal frequencies and normal modes of oscillation. Describe each mode.

A particle of mass m is moving in one dimension in a field with potential energy

$$U(x) = U_0 \left[2 \left(\frac{x}{a} \right)^2 - \left(\frac{x}{a} \right)^4 \right],$$

where U_0 and a are positive constants.

(a) Find the force $F(x)$, acting on the particle.

(b) Sketch $U(x)$. Find the positions of stable and unstable equilibria.

(c) What is the angular frequency ω of small oscillations about the stable equilibrium?

(d) What is the minimum speed the particle must have at $x = 0$ to escape to infinity?

(e) At $t = 0$, the particle is at $x = 0$ and its velocity is positive and equal in magnitude to the escape speed of part (d). Find $x(t)$ and sketch the result.

Baylor University Physics Ph.D. Program
Preliminary Exam 2010

Part II: Quantum Mechanics
Friday, August 13, 2010, 1:30 – 4:30pm

Problem 1

Variational Principle and Trial Wave Functions:

(a) Show that the lower limit to the energy expectation value of a trial wave function is the actual ground state energy of the system. (For ease, assume the potential has only bound, discrete states and no unbound, continuous states.)

(b) Use the variational principle to estimate the ground state energy of a particle in the potential

$$V(x) = \begin{cases} cx, & x > 0 \\ \infty, & x \leq 0 \end{cases}$$

where c is a constant.

Take

$$\psi(x) = \begin{cases} xe^{-ax}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

as the trial function.

Problem 2

A particle of mass m is bound in a modified one-dimensional square well defined by the potential energy function:

$$V(x) = \begin{cases} \infty & (x < 0) \\ 0 & (0 < x < a) \\ V_0 & (x > a) \end{cases}$$

(a) Which forms correspond to the bound-state solutions of the energy eigenvalue equation in each of the regions defined above?

$$
\begin{aligned}
u(x) &= A \sin kx + B \cos kx & (1) \\
u(x) &= C e^{\gamma x} + D e^{-\gamma x} & (2) \\
u(x) &= E & (3)
\end{aligned}
$$

Also, express k and γ with V_0, E, m and $\hbar$.

(b) What conditions must be satisfied by these solutions at $x = \infty$, $x = 0$, and $x = a$?

(c) Show that the depth of the well V_0 must satisfy:

$$V_0 > \frac{\pi^2 \hbar^2}{8ma^2}$$

(d) The above potential model may be used describe the attraction between a nucleus of radius 5 fm and a neutron of mass 940 MeV/c^2. Calculate (in MeV) how deep the well must be in order to bind the neutron ($\hbar c$ = 197 MeV·fm).

2

The Hamiltonian operator for a three-state system is given by,

$$H = H_0 \Big(|1\rangle \langle 1| - i \, |2\rangle \langle 3| + i \, |3\rangle \langle 2| \Big),$$

where H_0 is a real constant.

(a) Find the eigenvalues and the corresponding eigenkets of H.

(b) If an operator A is given by

$$A = A_1 \, |1\rangle \langle 1| + A_2 \, |2\rangle \langle 2| + A_3 \, |3\rangle \langle 3|$$

find the conditions so that A and H can share the same eigenkets, and check if the eigenkets of H found in (a) are also eigenkets of A.

Problem 4

Consider a system of two particles, with spins $j_1 = 2$ and $j_2 = 1/2$, respectively.

(a) How many different spin states can the system have? Label them first in terms of $|j, m\rangle \equiv |j_1 j_2; jm\rangle$ and then in terms of $|m_1, m_2\rangle \equiv |j_1 j_2; m_1 m_2\rangle$.

(b) Express all of the six states, $\left| j = \frac{5}{2}, m \right\rangle \left(m = \frac{5}{2}, \frac{3}{2}, ..., -\frac{5}{2} \right)$, in terms of $|m_1, m_2\rangle$, where you may use the relation,

$$J_- |j, m\rangle = \sqrt{(j + m)(j - m + 1)} \, |j, m - 1\rangle,$$

with $J_- = J_{1-} \oplus J_{2-}$.

Consider a quantum mechanical harmonic oscillator with Hamiltonian, in dimensionless units, given by

$$H = \frac{1}{2}\left(p^2 + q^2\right)$$

where q is the position coordinate and p is the respective conjugate momentum. If $|n\rangle$ is the energy eigenstate with energy eigenvalue $n + \frac{1}{2}$ in dimensionless units, show that the Heisenberg uncertainty product in this state is

$$\Delta q \Delta p = n + \frac{1}{2}.$$

Here we recall the uncertainty of O in the state $|A\rangle$ is

$$(\Delta O)^2 = \left\langle A|O^2|A\right\rangle - \langle A|O|A\rangle^2 .$$

Problem 6

Given the one-dimensional delta-function potential,

$$V(x) = -\lambda\delta(x-a),$$

as well as the Schrodinger equation,

$$\left[-\frac{\hbar}{2m}\frac{d^2}{dx^2} + V(x)\right]u(x) = \frac{\hbar^2 k^2}{2m}u(x),$$

where $E = \frac{\hbar^2 k^2}{2m}$, show that for an initial plane wave $\sim e^{ikx}$ moving in the $+x$-direction, the reflection, $r(k)$, and transmission, $t(k)$, amplitudes are given by

$$r(k) = \frac{im\lambda}{\hbar^2 k}\frac{e^{2ika}}{\left(1 - i\frac{m\lambda}{\hbar^2 k}\right)}$$

$$t(k) = \frac{1}{\left(1 - i\frac{m\lambda}{\hbar^2 k}\right)}$$

Baylor University Physics Ph.D. Program
Preliminary Exam 2010

Part III: Electricity & Magnetism
Saturday, August 14, 2010, 9:00 am – 12:00pm

$\boxed{\text{Problem 1}}$

For a single charge q, the rate of doing work by external fields $\vec{B}$ and $\vec{E}$ is

$$q\vec{v} \cdot \vec{E},$$

where $\vec{v}$ is the velocity of the charge.

(a) Find the corresponding expression for a continuous distribution of charge and current and interpret it physically.

(b) Use Maxwell's equations to express the result from part (a) in terms of the fields alone.

(c) From your results in part (b), verify Poynting's theorem

$$\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = -\vec{J} \cdot \vec{E}$$

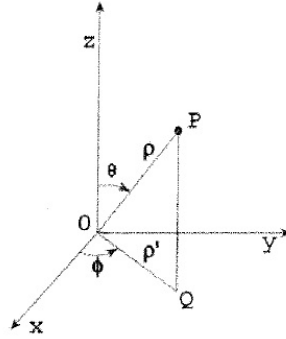Find expressions for the terms u and $\vec{S}$. Interpret those terms and the physical significance of the theorem.

(a) Using the method of images, find the electric field produced by a point charge placed in front of an infinite grounded conducting plane.

(b) Assume a thundercloud can be modeled as an electric dipole, whose axis is vertical and is held stationary above the ground. Show that the electric field observed at a point on the ground is proportional to

$$3 \sin^5 \alpha - \sin^3 \alpha$$

where α is the elevation angle of the cloud (the angle of the vector pointing from the observation point to the center of the dipole, measured with respect to the ground).

Problem 3

(a) Write down Helmhotz' equation in spherical coordinates.

(b) Assume a separated solution of the form $u = R(r)Y(\theta, \phi)$ to find equations

$$\frac{1}{Y}\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 Y}{\partial\phi^2}\right] = -\lambda$$

and

$$\frac{1}{rR}\frac{d^2}{dr^2}(rR) + k^2 - \frac{\lambda}{r^2} = 0$$

(c) Show that for Laplace's equation ($k^2 = 0$), the solution to the radial equation above can be written in the form $R = Ar^{\alpha_1} + Br^{\alpha_2}$, where $\alpha = \frac{1}{2}(-1 \pm \sqrt{1 + 4\lambda})$ and that the solution to the angular equation above can be written in the form $\phi = Ae^{im\varphi} + Be^{-im\varphi}$ and $P = CP_l^m(x) + DQ_l^m(x)$, where $l(l+1) = \lambda$.

(d) Employ the above to find the general solution $\Psi(r)$ to Laplace's equation (where $\Psi(r)$ is now the electrostatic potential) for a conducting sphere in a uniform external electric field with boundary conditions given by
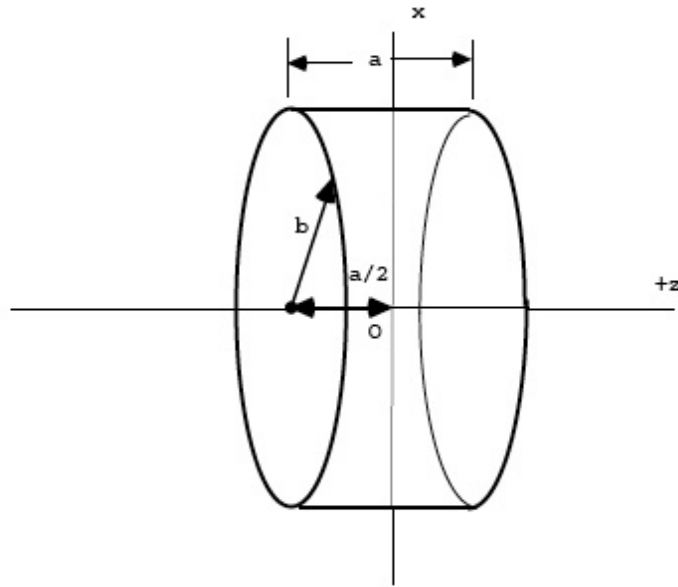
$$\Psi(a, \theta, \varphi) = \text{const}$$
$$\Psi(r \to \infty, \theta, \varphi) \longrightarrow -E_0\, r \cos\theta$$

where E_0 is the external electric field, assumed to be in the z direction:

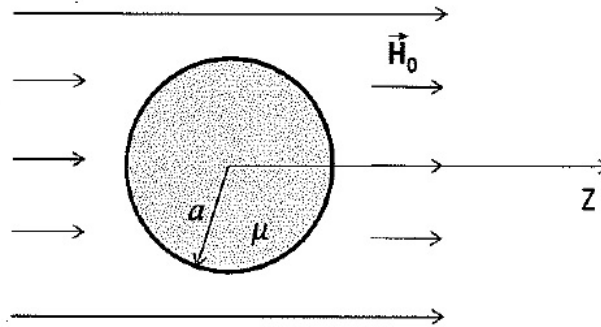$$-\nabla(-E_0\, r \cos\theta) = \nabla E_0\, z = E_0\, \hat{k}$$

3

Find the electric field E_z along the axis of symmetry (z) of a uniform cylinder of radius b and thickness a with constant charge density ρ. Show that $E_z = \text{const} \cdot z$ near the geometric center, at the origin O. Find the constant. [Hint: Try calculating E_z directly, not the potential.]

Problem 5

A sphere with a radius a, and of material having magnetic permeability μ, is placed in an uniform magnetic field $\vec{H}_0(\vec{B}_0 = \mu_0 \vec{H}_0)$.

(a) Working in spherical coordinates (r, θ, ϕ), write down the equations and boundary conditions which will determine the magnetic scalar potential inside and outside the sphere.

(b) Solve these equations and find the magnetic scalar potential inside and outside the sphere.

(c) Use your results to find the magnetic induction $\vec{B}$, magnetic field $\vec{H}$, and magnetization $\vec{M}$, inside the sphere.

Useful relations:

$$
\begin{aligned}
P_0(\cos\theta) &= 1 \\
P_1(\cos\theta) &= \cos\theta \\
\int_{-1}^{+1} P_l(x)P_{l'}\, dx &= \frac{2\delta_{l,l'}}{2l+1} \\
\frac{\partial P_l(\cos\theta)}{\partial\theta} &= P_l^1
\end{aligned}
$$

5

Problem 6

Maxwell Equations

(a) Write down the macroscopic Maxwell equations in terms of free charge ρ and free volume current density $\vec{j}$.

(b) Show that Maxwell equations are consistent with the conservation of electric charge.

(c) Write down the Maxwell equations in vacuum in terms of free charge ρ and free volume current density $\vec{j}$. Using the Lorentz condition, show that Maxwell's equations in a vacuum can be written as inhomogeneous wave equations in terms of the vector and scalar potentials $(\vec{A}, \varphi)$.

Problem 1

The partition function of a system is given by:

$$Z = e^{aT^3V}.$$

Determine the system's

(a) Helmholtz free energy

(b) pressure

(c) entropy

(d) internal energy

Problem 2

Electromagnetic radiation at temperature T fills a cavity of volume V. If the volume is expanded quasi-statically to $64V$ while the radiation exchanges no heat with its surroundings, what is the final temperture T_f? Make your reasoning clear. You may use what you remember of the general properties of blackbody radiation, i.e. $P \propto U$.

A system has access to four energy levels: $-E$, $-3E$, E, and $3E$. Write the expression for the Helmholtz free energy of this system for occupation by (a) 3 bosons and (b) 3 fermions.

Problem 4

Determine the nature of the 3 singularities in:

$$F(z) = \frac{ze^{iz}}{(z^2 + a^2)}$$

and evaluate the residues for $a > 0$.

The wave equation describing the transverse vibration of a stretched membrane under a tension T and a uniform surface density ρ is given by

$$T\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \rho\frac{\partial^2 u}{\partial t^2}.$$

Using the variable separation method, find the general solution of the membrane stretched on a frame of length a and width b, and show that the natural angular frequencies of such a membrane are given by

$$\omega^2 = \frac{\pi^2 T}{\rho}\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right),$$

where n and m are integers.

Problem 6

Using a Laplace transform, solve the problem $x'' - 4x' + 4x = h(t)$, with $x(0) = 2$ and $x'(0) = 0$, where

$$h(t) = \begin{cases} 0, & t < 1 \\ 6, & t > 1 \end{cases}$$

Problem 1.

A forest-fire-fighting airplane gliding horizontally at 200 km/hr lowers a scoop to load water from a lake. It continuously picks up 1/10 of the airplane's initial mass in water every 10 s.

a) Neglecting friction, find an expression for the airplane's speed as a function of time.
b) What is its speed after 10 s?
c) With a frictional force $F = -bv$ and a constant time rate of loading, find and solve the equation of motion of the airplane during the process.
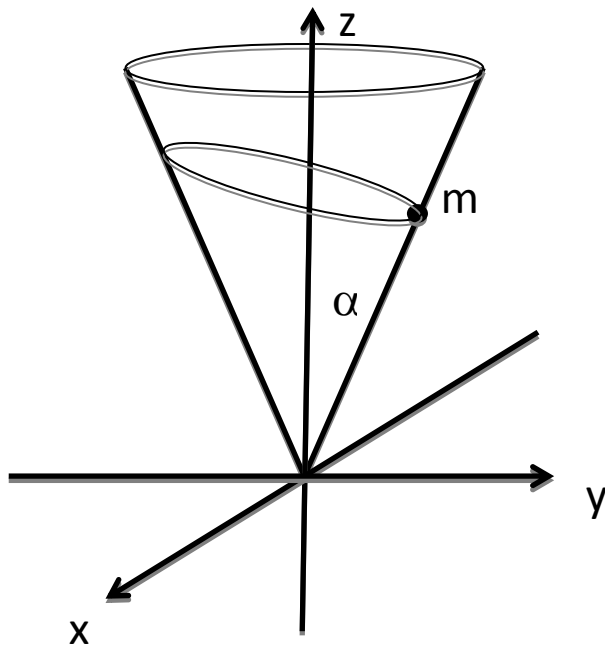
Problem 2.

A straight thin wire with one end at the origin rotates in the xy-plane with constant angular velocity ω. A bead of mass m slides along the wire without friction. At time $t = 0$, the bead is located on the +y axis at a distance r_0 from the origin and the component of the bead's velocity in the radial direction is zero. Ignore gravity.

a) Find $\theta(t)$, the angular position of the bead at time t.
b) Find $r(t)$, the radial position of the bead at time t.
c) Construct the Hamiltonian for the system and determine whether it is a constant.
d) Is the total energy of the system conserved? Justify your answer.
e) Explain how your answers to c) and d) are consistent.

Problem 3.

A particle of mass m is confined to move on the frictionless surface of a right circular cone whose axis is vertical, with a half opening angle α. The vertex of the cone is at the origin and the axis of symmetry is the z axis. For a given non-zero angular momentum L about the z-axis, find:

a) the height z_0 at which one can have a uniform circular motion in a horizontal plane.
b) the frequency of small oscillations about the solution found in part a). Give your answer in terms of only m, α, L, and the acceleration due to gravity g.
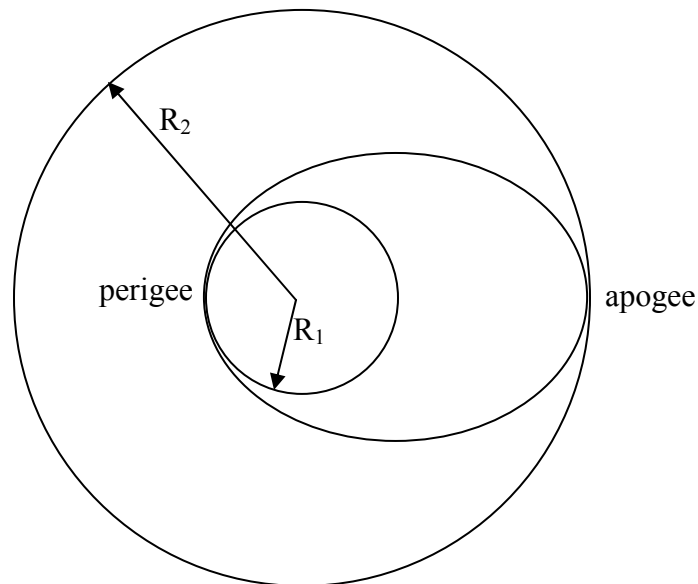
Problem 4.

The most efficient way to transfer a spacecraft from an initial circular orbit at R_1 to a larger circular orbit at R_2 is to insert it into an intermediate elliptical orbit with radius R_1 at perigee and R_2 at apogee. The following equation relates the semi-major axis (a), the total energy of the system (E) and the potential energy $U(r) = -GMm/r \equiv -k/r$ for an elliptical orbit of the spacecraft (mass: m) about the earth (mass: M):

$$R_1 + R_2 = 2a = \frac{k}{-E}.$$

a) Derive the relation between the velocity v and the radius R for a circular orbit.
b) Determine the velocity increase required to inject the spacecraft into the elliptical orbit as specified by R_1 and R_2. Let v_1 be the speed in the initial circular orbit and v_p be the speed at perigee after the first boost, so that the velocity increase is $\Delta v = v_p - v_1$.
c) Determine the velocity increment required to insert the spacecraft into the second circular orbit when it reaches apogee at $r = R_2$. Let v_2 be the speed in the final orbit and v_a be the velocity at apogee so $\Delta v = v_2 - v_a$.

Problem 5.

Time derivatives in a rotating reference frame are related to time derivatives in an inertial frame by

$$\left(\frac{d\vec{\mathbf{r}}}{dt}\right)_{inertial} = \left(\frac{d\vec{\mathbf{r}}}{dt}\right)_{rotating} + \Omega \times \vec{\mathbf{r}}$$

where Ω is the angular velocity of the rotating system.

a) By taking derivatives of the position vector **r**, derive the relationship between the acceleration in a rotating frame and the acceleration in an inertial frame. Identify the "Coriolis" and "centrifugal" terms in this expression.
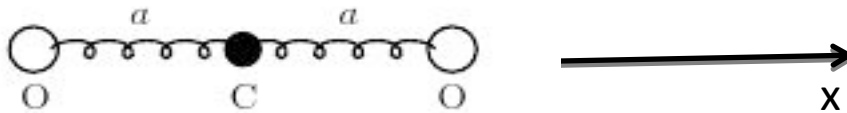
b) On a rotating earth, the centrifugal force causes a plumb bob to be deflected slightly away from a line pointing directly to the center of the earth. This direction defines the direction of the "effective" gravitational acceleration, $\vec{\mathbf{g}}_e$, which is a combination of local gravity $\vec{\mathbf{g}}$ and the centrifugal force. Determine the location and magnitude of the maximum difference between $\vec{\mathbf{g}}$ and $\vec{\mathbf{g}}_e$. What is the angular deflection at this latitude? (The radius of the earth is 6.37×10^6 m.)

Problem 6.

A CO_2 gas molecule is linear, as shown in the figure below, with its long molecular axis in parallel with the x-axis. The equilibrium distance between the carbon atom and each of the two oxygen atoms is a. The spring constant of each CO bond is k. The mass of an oxygen atom is M, and the mass of a carbon atom is m.

a) Find the frequencies of all of the normal modes of the molecule that have motion only along the x-axis,

b) Describe the motions of the atoms for each normal mode.

Ignore the sizes of the atoms.

Problem 1.

Recall that the raising and lowering operators for the harmonic oscillator are defined by

$$a^\dagger = \left(\frac{K}{2}\right)^{1/2} x - \frac{\hbar}{(2m)^{1/2}} \frac{\partial}{\partial x}$$

$$a = \left(\frac{K}{2}\right)^{1/2} x + \frac{\hbar}{(2m)^{1/2}} \frac{\partial}{\partial x} \ .$$

(a) Using the fact that the position coordinate variable x is self-adjoint, and the derivative operator $\partial/\partial x$ is anti-self-adjoint, show that the raising operator $a^\dagger$ is the adjoint of the lowering operator a.

(b) Show that the Hamiltonian is self-adjoint for a one-dimensional harmonic oscillator, where

$$H = a^\dagger a + \hbar\omega/2 = aa^\dagger - \hbar\omega/2.$$

(c) Let the wave function of the mth energy level $E_m = (m + 1/2)\hbar\omega$ in a harmonic oscillator be $\psi_m(x)$, where $\langle\psi_n|\psi_m\rangle = \delta_{nm}$. The raising operator yields

$$a^\dagger\psi_n = N\psi_{n+1}.$$

Find N, assuming N is real and positive.

(d) Evaluate $\langle\psi_n|x\psi_m\rangle$.

(e) Evaluate $\langle\psi_n|p\psi_m\rangle$, where $p = -i\hbar\partial/\partial x$.

Problem 2.

Consider a quantum system whose Hamiltonian admits just two eigenstates, ψ_a (with energy E_a), and ψ_b (with energy E_b). These states are orthogonal, normalized, and nondegenerate with $E_a < E_b$. states. Next, a perturbation H' is introduced, having the following matrix elements:

$$\langle \psi_a \,|\, H' \,|\, \psi_a \rangle = \langle \psi_b \,|\, H' \,|\, \psi_b \rangle = 0 \text{ and } \langle \psi_a \,|\, H' \,|\, \psi_b \rangle = \langle \psi_b \,|\, H' \,|\, \psi_a \rangle = h.$$

a. Find the exact eigenvalues of the perturbed Hamiltonian.

b. Using perturbation theory, calculate the energies of the perturbed system to second order.

c. Estimate the ground state energy of the perturbed system using the variational principle, with a trial function of the form

$$\psi = (\cos \phi)\psi_a + (\sin \phi)\psi_b,$$

where ϕ is an adjustable parameter.

Simplify your solution by eliminating the trig functions in the energy expression, and then compare with your answers in parts **a.** and **b.** Is your answer consistent?

Suggestion: Define $\varepsilon = 2h/(E_b - E_a)$ for ease in simplifying your expressions. Substitute back in for ε once your solution is simplified.

Problem 3.

Consider the four-state system consisting of two non-identical spins. Hence all states can be written as a linear combination of the four orthonormal states

$$|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$$

where the arrows refer to the direction of the spin in the z-direction. Suppose that the Hamiltonian is given by

$$H = \lambda P_{12}$$

Where P_{12} is the operator that exchanges the first spin with the second spin and $\lambda > 0$.

a) Find the normalized eigenstates and eigenvalues of the Hamiltonian.

b) Suppose that at time $t = 0$ the system is in state $|\uparrow\downarrow\rangle$. Find the probability as a function of t and λ that a measurement of the z-component of the first spin will be $+\hbar/2$.

c) Again suppose that at time $t = 0$ the system is in state $|\uparrow\downarrow\rangle$. Find the probability as a function of t and λ that a measurement of the x-component of the first spin will be $+\hbar/2$.

d) Now assume that the spins are identical fermions. Which of the energy eigenstates, if any, are allowed?

Problem 4.

A particle of mass, m, moves in three dimensions in the potential

$$V(\vec{r}) = F(r) + \beta z,$$

where $F(r)$ is a function of the radial distance, r, from the origin, β is a real constant, and z is the z component of the position vector, $\vec{r}$.

(a) Show that $F(r)$ commutes with L_x, the x component of the orbital angular momentum observable.

(b) Show that L_z, the z component of the orbital angular momentum observable, is conserved.

(c) Find the time rate of change of the expectation value of L_x.

Problem 5.

a) Determine the energy levels and normalized wave functions of a particle in a 1-dimensional potential box. The potential energy of the particle is

$$V = \infty \text{ for } x < 0 \text{ and } x > a, \text{ and } V = 0 \text{ for } 0 < x < a.$$

b) Show that a particle in this potential box has an expectation value <x> that is independent of the energy level n and determine <x>. Also show that

$$\langle (x - \langle x \rangle)^2 \rangle = a^2/12 \, [\, 1 - 6/(n\pi)^2 \,].$$

c) Determine the uncertainty σ of the momentum p of a particle in the nth energy state in this potential box.

Possible integrals of use:

$$\int x^m \sin^n x \, dx = \frac{x^{m-1}\sin^{n-1}x}{n^2}\{m \sin x - nx \cos x\} + \frac{n-1}{n}\int x^m \sin^{n-2} x \, dx - \frac{m(m-1)}{n^2}\int x^{m-2}\sin^n x \, dx$$

$$\int x^m \cos^n x \, dx = \frac{x^{m-1}\cos^{n-1}x}{n^2}\{m \cos x + nx \sin x\} + \frac{n-1}{n}\int x^m \cos^{n-2} x \, dx - \frac{m(m-1)}{n^2}\int x^{m-2}\cos^n x \, dx$$

Problem 6.

Use the Heisenberg equation of motion and the relation

$$(\Delta A)^2(\Delta B)^2 = \langle u | \hat{A}^2 | u \rangle \langle | \hat{B}^2 | u \rangle \geq |\langle u | \hat{A}\hat{B} | u \rangle|^2$$

To derive the uncertainty relationship for energy and time.

Problem 1.

a) Find the *fully relativistic* formula for the radius of a circular orbit, R, of a particle with charge, q, in a uniform magnetic field, $\vec{B}$, with momentum, $\vec{p}$.

b) Show that for pure planar motion that the energy of the particle is

$$E = mc^2 \sqrt{1 + \frac{R^2 q^2 B^2}{c^2}}.$$

Problem 2.

Starting from the Biot-Savart Law,

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \vec{\nabla} \times \int \frac{\vec{J}(x')}{|\vec{x} - \vec{x}'|} d^3 x'$$

a) Show that, in steady state, magnetic induction satisfies

$$\vec{\nabla} \cdot \vec{B} = 0 \text{ and } \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}.$$

b) For the time dependent field, together with the continuity equation, show that the magnetic induction satisfies the Maxwell-Ampere law:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}.$$
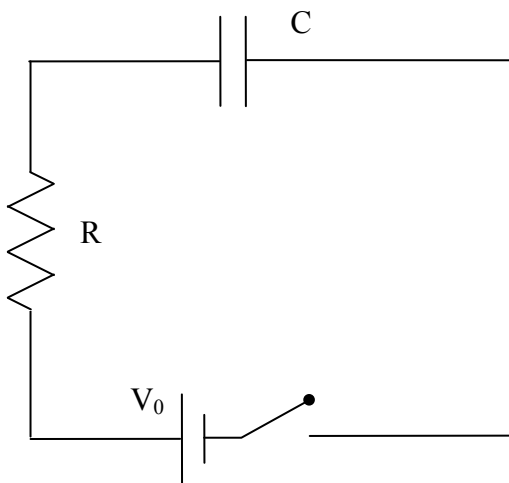
Problem 3.

A circular parallel plate capacitor of radius a and plate separation d is connected in series with a resistor R and a switch, initially open, to a constant voltage source V_o.

a) The switch is closed at time $t = 0$. Find an expression for the strength of the magnetic field (B) between the plates as a function of time and show that it could be written as

$$B = \frac{\mu_o V_o r}{2\pi a^2 R} e^{-t/RC}$$

where C is the capacitance and r is the radial distance from the line joining the two centers of the parallel plates.

b) What is the maximum electric energy stored in the capacitor?

c) What is the maximum magnetic energy stored in it?

d) If the time constant of the magnetic field is very long compared to a/c (c, speed of light) and $d \ll a$, show that the stored magnetic energy is less than 12.5% of stored electric energy.

Problem 4.

Show that the magnetic induction at a point P with coordinate $\vec{x}$ produced by a closed current loop carrying current I is

$$\vec{B} = \frac{\mu_o I}{4\pi} \vec{\nabla}\Omega,$$

where Ω is the solid angle subtended at P by the loop surface so that Ω is positive (negative) according as $\hat{n}$ points away (toward) P when $\hat{n}$ is the unit normal to the surface spanning the loop with $\hat{n}$ defined by the direction of current flow using the right-hand rule.

Problem 5.

Consider a corner geometry in cylindrical coordinates. The electrostatic potential $\Phi(\rho,\phi)$ in the region $\{0 \leq \rho \leq \rho_0,\ 0 \leq \phi \leq \beta\}$ satisfies the boundary conditions

$$\Phi(\rho, 0) = \Phi(\rho, \beta) = V,$$
$$\Phi(\rho_0, \phi) = V/2.$$

Find the limiting behavior of the electric field components E_ρ, E_ϕ for $\rho \to 0$ for $0 \leq \phi \leq \beta$.

Problem 6.

Consider a hollow rectangular wave guide made of a perfect conducting material with height a in the x-direction and width b in the y-direction ($a > b$). Assume that monochromatic waves propagate down the guide, so that **E** and **B** have the generic forms

$$\tilde{E}(x,y,z,t) = \tilde{E}_0(x,y)e^{i(kz-\omega t)}$$
$$\tilde{B}(x,y,z,t) = \tilde{B}_0(x,y)e^{i(kz-\omega t)}.$$

a) Using Maxwell's equations, show that in the absence of charges and currents, $\tilde{E}$ and $\tilde{B}$ satisfy the wave equation

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

where v is speed of propagation of the waves.

b) By solving the wave equation subject to the boundary condition $\frac{\partial f}{\partial n} = 0$, where n is the direction normal to the surface, find functions for $\tilde{E}_{0z}$ and $\tilde{B}_{0z}$ for the condition of propagation of TE (transverse electric) waves.

c) Find an expression for the cutoff frequencies ω_{mn} and determine the lowest cutoff frequency.

Baylor University Physics Ph.D. Program
Preliminary Exam 2011

Part IV: Statistical Mechanics, Mathematical Physics
Saturday, August 13, 2010, 1:30 pm – 4:30 pm

Problem 1.

The probability distribution of the momentum of molecules with mass m at temperature T can be written as

$$f(\bar{p}) = \frac{1}{(2\pi\sigma)^{3/2}} e^{-p^2/2\sigma^2}.$$

a) Use translational kinetic energy to prove that

$$\sigma^2 = mkT.$$

b) Show that the speed distribution of molecules can be written as

$$f(u) = \frac{4}{\sqrt{\pi}} \left(\frac{m}{2kT}\right)^{3/2} u^2 e^{-mu^2/2kT}.$$

c) A diatomic gas is contained in a vessel from which it leaks through a fine hole. The distribution of speeds of molecules which are incident on the hole is $u\, f(u)$ since the faster molecules arrive at the hole more quickly than the slower ones. Show that the average kinetic energy of the molecules leaving through the hole is $2kT$.

Problem 2.

For relativistic particles, the energy E of a particle with rest mass m and momentum $p = \hbar k$ is given by the following relationship:

$$E^2 = p^2 c^2 + m^2 c^4.$$

An accelerator is used to bring a stream of electrons to relativistic speeds in a beam that is essentially one electron wide.

a) In the extreme relativistic limit, determine the density of states and, with that, the partition function for a particle in that beam.

b) The beam of N electrons is allowed to strike a target surface. Find an expression for the pressure as a function of temperature that the momentum transfer causes as the beam strikes the target surface.

Problem 3.

Consider an ideal gas (non-interacting and non-relativistic) in a container V. It is composed of N "red" atoms of mass m, N "blue" atoms of mass m, and N "white" atoms of mass m. Atoms of the same color are indistinguishable. Atoms of different color are distinguishable.

a) Calculate the partition function of this gas.

b) Then, using the partition function, calculate the entropy of the gas.

c) Compare the entropy of this mixture with that of $3N$ "red" atoms (i.e. pure gas). Does it differ from that of the mixture? If so, by how much?

Problem 4.

a) Calculate the Fourier transform of the function

$$e^{-ax^2} \text{ for } a > 0.$$

b) Use the result you obtain to verify the following representations of the distribution

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{e^{-x^2/\varepsilon^2}}{\varepsilon\sqrt{\pi}},$$

and

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \frac{\varepsilon}{\varepsilon^2 + x^2}.$$

Problem 5.

Use the Green function to solve the following equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

satisfying the boundary conditions: $u(0, y) = u(x, b) = 0$, $u(\infty, y) < \infty$.

Show that the solution of this equation is given by

$$u(x, y) = \int_0^\infty \int_0^b G(x \,|\, \xi; y \,|\, \eta) f(\xi, \eta) d\xi d\eta$$

where

$$G(x|\xi; y|\eta) = \begin{cases} -\dfrac{2}{\pi} \displaystyle\int_0^\infty \dfrac{\sin k\xi \sin kx \sinh ky \sinh k(b-\eta)}{k \sinh kb} dk, (y < \eta) \\[4mm] -\dfrac{2}{\pi} \displaystyle\int_0^\infty \dfrac{\sin k\xi \sin kx \sinh k\eta \sinh k(b-y)}{k \sinh kb} dk, (y > \eta) \end{cases}.$$

Problem 6.

a) Find the general solutions of the equation $\dfrac{F''(x)}{F(x)} = \lambda$, where λ is a constant.

b) Find all solutions of the homogeneous wave equation $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial t^2} = 0$.

c) Find all solutions that satisfy the conditions

$$u(0,t) = u(3,0) = 0.$$

d) Find all solutions of the equation $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial t^2} = 0$ that satisfy the conditions

$$u(0,t) = u(3,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0 \ .$$

e) Find the unique solution of the equation $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial t^2} = 0$ that satisfies the conditions

$$u(0,t) = u(3,0) = 0, \quad u(x,0) = x, \quad \frac{\partial u}{\partial t}(x,0) = 0 \ .$$
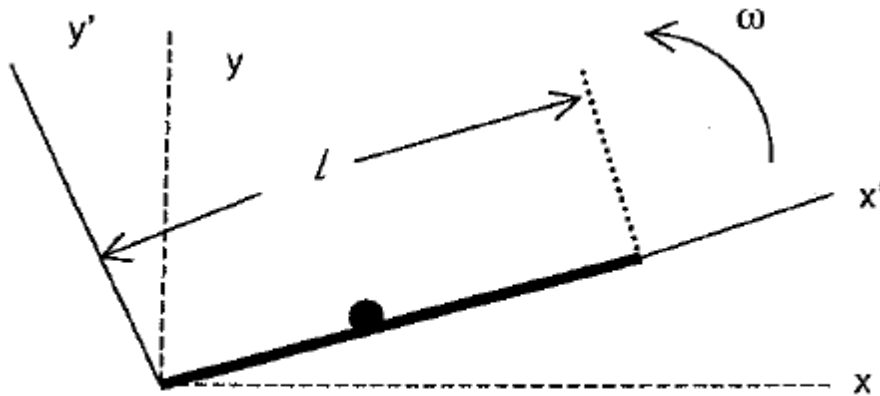
Problem 1.

A smooth rod of length l rotates in a plane with constant angular velocity ω about an axis fixed at the end of the rod and perpendicular to the plane of rotation. A bead of mass m is initially positioned at the stationary end of the rod and given a slight push such that its initial speed directed along the rod is $\mathcal{E} = \omega\, l$.

    (a) Show that the time t it takes for the bead to reach the other end of the rod is:

$$t = \frac{1}{\omega} \sinh^{-1}(1)$$

    (b) Find the reaction force that the rod exerts on the bead as experienced by the bead.

Problem 2.

An object of mass, m, with initial velocity, v_0, enters a cloud of gas. The gas exerts a drag force on the object that is proportional to the nth power of the object's velocity, such that

$$F = -m\beta v^n$$

with $n > 0$. Assuming the object enters the cloud at $t = 0$ and $x = 0$:

(a) For what values of n will the mass stop after a finite distance?
(b) Find the position of the object as a function of time for the n=2 case.
(c) Expand the result from part (b) for small t and comment on the difference between first and second order in the expansion.
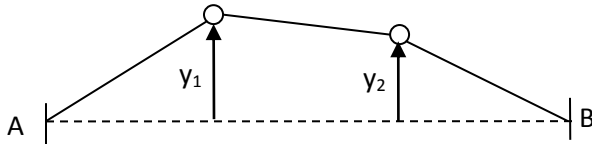
Problem 3.

A rocket of rest length L_0 is moving with constant speed v along the z axis in the +z direction in an interial system. An observer on the z axis observes the apparent length of the approaching rocket at any time by noting the z coordinates that can be seen for the head and tail of the rocket.

   (a) Find the z coordinate of the tail as seen by the observer at time t_0 in the observer's reference frame.
   (b) Find the z coordinate of the head as seen by the observer at time t_0 in the same reference frame.
   (c) Determine the length of the rocket as observed by the observer
   (d) Interpret your answer to part (c).

Problem 4.

Two equal masses move on a frictionless horizontal table. They are held by three identical taut
strings (each of length L, tension T), as shown in the figure, so that their equilibrium position
is a straight line between the anchors at A and B. The two masses move in the transverse (y)
direction, but not in the longitudinal (x) direction. Write down the Lagrangian for the small
displacements (y_1, $y_2 \ll L$), and find and describe the motion in the corresponding normal
modes. (Hint: the potential energy in each string is Td, where d is the distance the string
stretches.)

Problem 5.

A heavy symmetrical top with one point fixed is spinning in a constant gravitational field. The mass of the top is m and the distance from the center of mass to the point of contact is l.

Obtain from Euler's equations of motion the condition

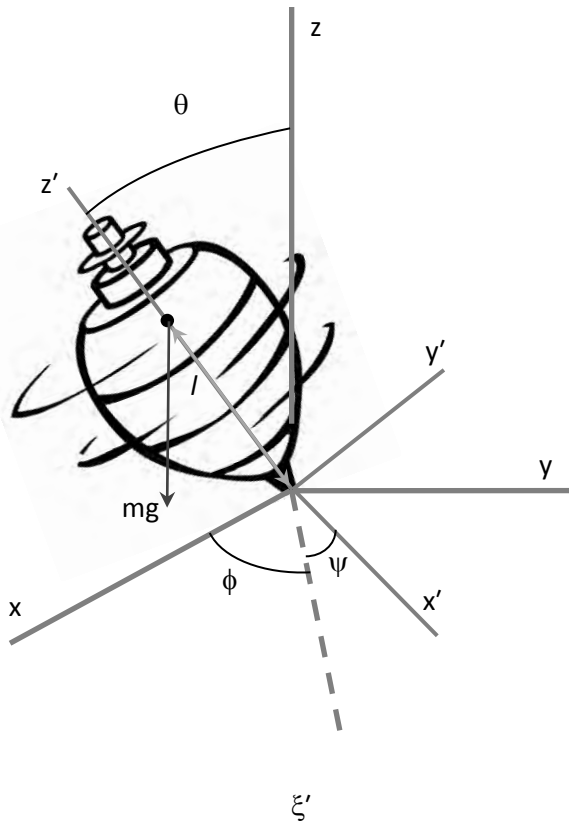$$mgl = \dot{\varphi}[I_3\omega_3 - I_1\dot{\varphi}\cos\theta]$$

for the uniform precession of the symmetrical top, by imposing the requirement that the motion be a uniform precession ($\dot{\varphi}$ = const) without nutation ($\dot{\theta} = 0$).

The components of $\boldsymbol{\omega}$ with respect to the body axes are

$$\omega_{x'} = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$

$$\omega_{y'} = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$

$$\omega_{z'} = \dot{\varphi}\cos\theta + \dot{\psi}$$

Problem 6.

Consider a particle moving in one dimension with Lagrangian

$$\mathcal{L} = \frac{m}{2} \left(\dot{x}^2 - \omega^2 x^2 \right) e^{\gamma t},$$

where ω and γ are positive, real constants.

(a) Find the equation of motion for the particle. It should look familiar – what sort of physical system does it describe?

(b) Find the canonical momentum, and from that, the Hamiltonian function. Is the Hamiltonian a constant of the motion? Is energy conserved? Explain.

(c) Perform the point transformation

$$s = x\, e^{\gamma t/2}$$

and determine the equation of motion for s.
Describe the solutions for $\gamma < 2\omega$ and for $\gamma > 2\omega$.

(d) Consider a canonical transformation of the form $x = x\,(X, P, t)$, $p = p\,(X, P, t)$ leading to the following relation,

$$p\dot{x} - \mathcal{H}(x, p, t) = P\dot{X} - \tilde{\mathcal{H}}(X, P, t) + \frac{\mathrm{d}}{\mathrm{d}t} F(x, X, t).$$

The new Hamiltonian $\widetilde{\mathcal{H}}\,(X,P,t)$ obeys Hamilton's equations in the variables X, P. Use Hamilton's principle to explain why the third term does not modify the form of Hamilton's equations.

Problem 1.

A particle has a properly normalized wave function

$$\psi(x) = \frac{1}{(\pi\Delta^2)^{1/4}} e^{-(x-a)^2/2\Delta^2}$$

(a) Calculate $\langle X \rangle$ and $\langle P \rangle$.

(b) Calculate $\langle X^2 \rangle$ and $\langle P^2 \rangle$.

(c) Calculate ΔX and ΔP, and comment on how the product $\Delta X \Delta P$ relates to the Heisenberg Uncertainty principle.

Problem 2.

Consider three independent angular momenta, $\vec{J}_i$, i = 1, 2, 3. Show that the quantum mechanical state $|0\rangle$ for which $(\vec{J}_1 + \vec{J}_2 + \vec{J}_3)\,|0\rangle = \vec{0}$ is given by:

$$|0\rangle = \sum_{m_1,m_2,m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} |j_1 m_1\rangle \otimes |j_2 m_2\rangle \otimes |j_3 m_3\rangle,$$

where $|j_i m_i\rangle$ is the eigenstate of $\{J_i^2, J_{iz}\}$ with
$J_i^2\,|j_i m_i\rangle = j_i(j_i+1)\,|j_i m_i\rangle$, and
$J_{iz}\,|j_i m_i\rangle = m_i\,|j_i m_i\rangle$, for $i = 1,\ldots,3$.

Here, Wigner's 3-j symbol is defined as follows:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1-j_2-m_3}}{\sqrt{2j_3+1}}\,\langle j_1 m_1 j_2 m_2\,|\,j_3 - m_3\rangle,$$

in the standard notation wherein $|j_1 m_1 j_2 m_2\rangle = |j_1 m_1\rangle \otimes |j_2 m_2\rangle$, for example.

Problem 3.

Consider two neutrinos, $|v_e\rangle$ and $|v_\mu\rangle$. Electron neutrinos (v_e) are produced in the sun; the missing solar neutrino" problem is that not enough of them are observed at the Earth! Here is one possible solution:

Suppose the neutrinos have (small) masses, but that the eigenfunctions of the Hamiltonian are not $|v_e\rangle$ and $|v_\mu\rangle$, but mixtures:

$$\begin{aligned} |v_1\rangle &= \cos\theta \, |v_e\rangle + \sin\theta \, |v_\mu\rangle & \text{mass } m_1 \\ |v_2\rangle &= -\sin\theta \, |v_e\rangle + \cos\theta \, |v_\mu\rangle & \text{mass } m_2 \end{aligned}$$

The energy eigenvalue for each of these states (with momentum $p \gg m_i$) is

$$E_i = \sqrt{p^2 + m_i^2} \approx p + \frac{m_i^2}{2p}$$

Consider a process that produces an electron neutrino at time $t = 0$. Show that the probability for the state to become a muon neutrino, v_μ, after travelling a distance L is:

$$\mathcal{P} = \frac{1}{2}\sin^2 2\theta \left\{ 1 - \cos\left[\frac{(m_2^2 - m_1^2)L}{2p\hbar} \right] \right\}$$

These muon neutrinos would not be observed by the same search for the electron neutrinos, thereby explaining the "missing" solar neutrinos.

Note that we are using units in which $c = 1$.

Problem 4.

The operator Q satisfies the two equations

$$Q^\dagger Q^\dagger = 0$$
$$QQ^\dagger + Q^\dagger Q = 1.$$

The Hamiltonian for the system is

$$H = \alpha Q Q^\dagger,$$

where α is a real constant.

(a) Show that H is self adjoint.
(b) Express H^2, the square of H, in terms of H.
(c) Find the eigenvalues of H allowed by the result from part (b).

Problem 5.

The operators $L_\pm$ are defined by $L_\pm = L_x \pm L_y$, and satisfy the relations:

$$L_\pm |j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\hbar |j, m \pm 1\rangle$$

where $|j, m\rangle$ are eigenkets of J^2 and J_z, that is,

$$J^2 |j, m\rangle = j(j+1)\hbar |j, m \pm 1\rangle \qquad \text{and}$$

$$J_z |j, m\rangle = m\hbar |j, m\rangle.$$

(a) For $j = 1$, write down all the possible eigenkets of J^2 and J_z.
(b) Use the relations above to calculate $L_x |j = 1, m\rangle$, and then find all the eigenkets of L_x in terms of $|j = 1, m\rangle$.
(c) A beam of particles with $j = 1$ prepared in an oven with randomized polarizations is moving along the y-axis and passes through a Stern-Gerlach magnet with its magnetic field $\vec{B}$ along the x-axis. Draw a graph and show how many separate beams are to be observed after passing through the device (and label them appropriately).
(d) Let the emerging beam with $m_x = 1$ pass through a second Stern-Gerlach magnet with its magnetic field $\vec{B}'$ along the z-axis. Into how many beams will this beam split?

Problem 6.

A particle of mass m moves under in a one-dimensional potential given by:

$$V(x) = \begin{cases} V_0, & |x| \le a, \\ 0, & a < |x| < L, \\ \infty, & |x| \ge L, \end{cases}$$

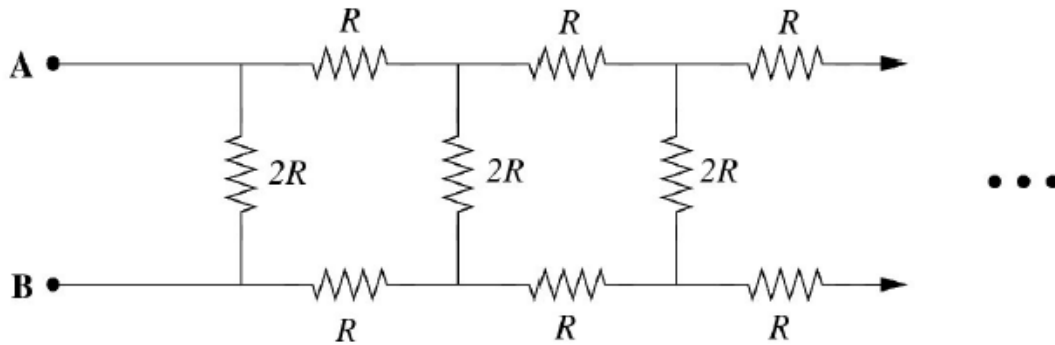where V0, a and L are positive constants with a < L.

(a) Assuming that the wave function has even parity, i.e., $\Psi_n(-x) = \Psi_n(x)$, find the energy spectra for $E_n < V_0$ and $E_n > V_0$ separately.
(b) Assuming that the wave function has odd parity, i.e., $\Psi_n(-x) = -\Psi_n(x)$, find the energy spectra for $E_n < V_0$ and $E_n > V_0$ separately.
(c) Consider the limit where a → 0, $V_0 \to \infty$, but the product aV_0 remains finite, say, $aV_0 = U_0/2$ where U_0 is a finite constant. Find the energy spectra for both the cases of even parity and odd parity as defined above.

Problem 1.

An infinite chain of resistors is shown in the figure below. Find the equivalent resistance between points A and B.

Problem 2.

If it were discovered experimentally that the electric field of a point charge q was proportional to $q\, r^{-2-\delta}\,\hat{r}$, where $\delta \ll 1$,

    a. Calculate $\vec{\nabla}\cdot\vec{E}$ and $\vec{\nabla}\times\vec{E}$ for $r\neq 0$,

    b. For this case, consider two concentric spherical shells of radii a and b (a > b) connected by a wire, with charge qa on the outer shell. Prove that:

$$q_b = \frac{-q_a\,\delta}{2\,(a-b)}\,[\,2b\ln(2a) + (a-b)\ln(a-b) - (a+b)\ln(a+b)] + O(\delta^2)$$

    c. Argue that measuring qb provides a mechanism (through determination of δ) for experimentally verifying the r^{-2} law.

Problem 3.

Consider a uniformly magnetized sphere of radius a centered about the origin $\vec{O}$. The magnetization $\vec{M}$ is given by

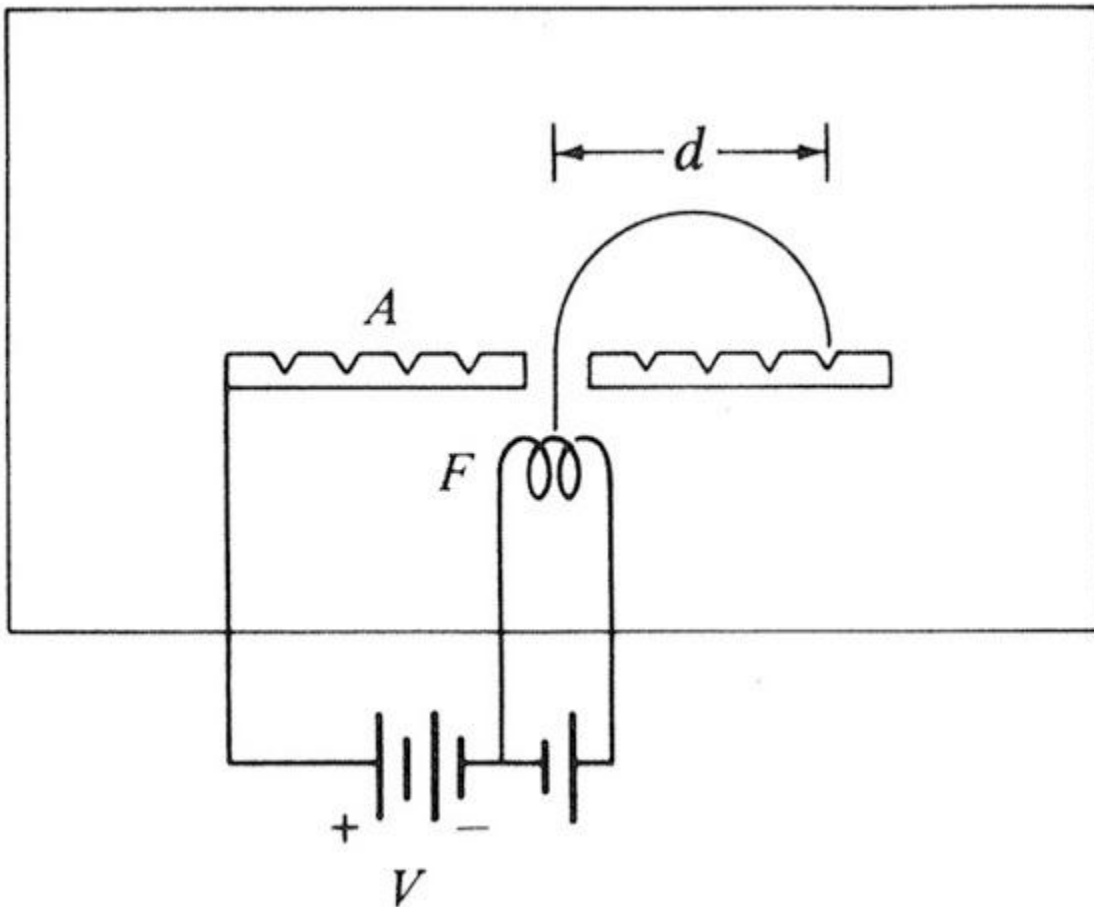$$\vec{M} = \begin{cases} M_0 \hat{x}, & r \leq a \\ \vec{0}, & r > a, \end{cases}$$

Where M_0 is a constant, $\vec{x} = (r, \theta, \phi)$ are the usual spherical coordinates, and $\hat{x}$ is the unit vector in the x direction. Find the magnetic field $\vec{H}$ everywhere.

Problem 4.

The charge-to-mass of the electron can be measured using a specially designed vacuum tube, as illustrated in the figure below. It contains a heated filament F and an anode A which is maintained at a positive potential relative to the filament by a battery of known voltage V. Electrons are released from the heated filament and are accelerated to the anode, which has a small hole in the center for the electrons to pass through into a region of constant magnetic field B, which points into the paper. The electrons then move in a semicircle of diameter d, hitting the detector as shown.
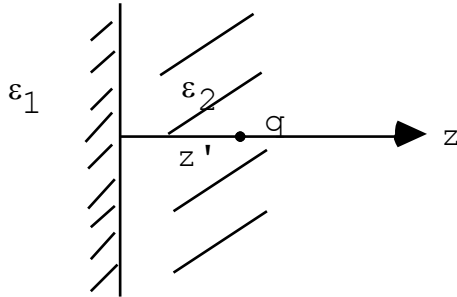
Prove that

$$\frac{e}{m_e} = \frac{aV}{(Bd)^2}$$

Problem 5.

A plane interface exists between two regions of unequal dielectric constant, ε_1 in the region $z < 0$ and ε_2 in the region $z > 0$. A point charge is located at $z' > 0$.
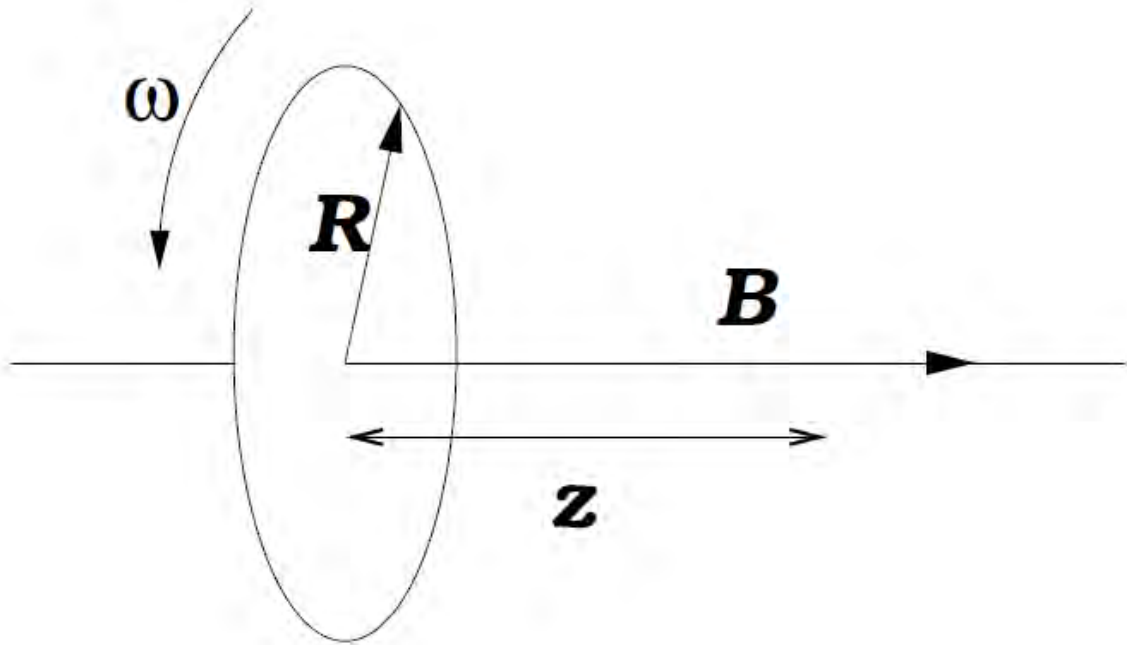

Find the electric potential everywhere.

Problem 6.

A thin, non-conducting disk of radius R is spinning around its symmetry axis with angular velocity $\vec{\omega}$. The disk is uniformly charged with a charge density per unit area σ.

(a) What is the exact expression for the magnetic field along the symmetry axis of the disk as a function of the distance, z, from the disk?

(b) For distances far from the disk, the disk looks like a magnetic dipole. What is the effective magnetic dipole moment?

(c) Show that the expressions in part (a) and (b) agree at large distances.

You may find the following to be useful:

$$\int \frac{r^3 dr}{(r^2+z^2)^{3/2}} = \left(\frac{r^2+2z^2}{\sqrt{r^2+z^2}}\right),$$

Biot-Savard formula: $d\vec{B} = \frac{\mu_0}{4\pi} \frac{id\vec{s} \times \vec{r}}{r^3}$,

Magnetic dipole moment of a N-loop coil: $\vec{\mu} = NI\vec{A}$,

Magnetic field by a magnetic dipole moment μ: $B = \frac{\mu_0}{2\pi} \frac{\mu}{(r^2+z^2)^{3/2}}$.

Part III: Statistical Mechanics, Mathematical Methods
Saturday, August 11, 2012, 1:30 p.m. – 4:30 p.m.

Problem 1.

For an interacting gas, the partition function can be written as

$$Q = \left(\frac{V - Nb}{N}\right)^N \left(\frac{mk_BT}{2\pi\hbar}\right)^{3N/2} e^{N^2a^2/Vk_BT}$$

where a and b are constants.

From the partition function

(a) Calculate the pressure and show that it is of the same form as the van der Waals equation.

(b) Calculate the internal energy.

(c) Is the internal energy of the interacting gas smaller or larger than that of the ideal gas?

Problem 2.

A thermodynamic ratchet can be thought of as a continuum of states:

$$E_i = \tfrac{1}{2} I \omega_i^2$$

where $\omega_i \geq 0$.

Determine:

   (a) $<U>$
   (b) $<\omega>$
   (c) $<C_v>$

from the canonical ensemble treatment of this system thermodynamically.

Problem 3.

The tension of an elastic rod is, at the temperature T, related to the length of the rod L by the expression $F = aT^2 (x - x_0)$, where a and x_0 are positive constants and F is the external force acting on the rod. The heat capacity at constant volume (i.e. constant length) of an un-stretched rod $(x = x_0)$, is given by $C_x = bT$, where b is a positive constant.

(a) Use the first law of thermodynamics and write down an expression for dS of the rod, where S is the entropy.

(b) Using S = S(T,x), show that $(\partial S / \partial x)_T = -2aT(x - x_0)$.

(c) If we know $S(T_0, x_0)$, calculate S(T, x) at any other T and x.

(d) If we stretch a thermally insulated rod from $x = x_I$ and $T = T_I$ to $x = x_F$ and $T = T_F$, calculate T_F in terms of T_I and other given parameters.

(e) Calculate $C_x = C_x (T, x)$ for an arbitrary length x, instead of for the length $x = x_0$.

Problem 4.

a. Find the Green function for the differential equation

$$y''(x) + y(x) = f(x)$$

on the interval $0 \leq x \leq \pi/2$, with $y(0) = y(\pi/2) = 0$.

b. Use your Green function from part **a.** to obtain a solution of the differential equation when $f(x) = \sin 2x$. Verify your solution by direct substitution.

Problem 5.

Find the first three nonzero terms of the Laurent expansion of the function

$$f(z) = tan(z) \text{ about } z = \pi/2.$$

Problem 6.

Consider the following integral,

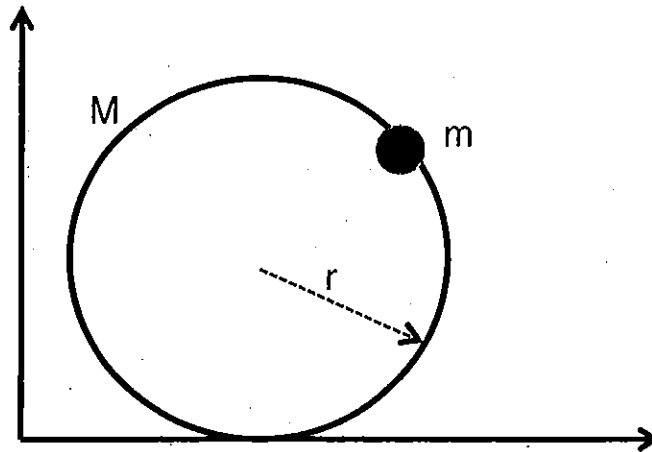$$I = \oint_C \frac{\sinh \pi z}{(z-1)^2 (z^2 + 1)} dz$$

where C is the circle $|z| = 2$ in the complex plane.

(a) Determine all the poles and their orders inside the circle $|z| = 2$.

(b) Calculate the integral, justify each step of your work.

(c) If C is not the circle $|z| = 2$, but any other closed curve that the circle $|z| = 2$ is included in, then determine the value of the integral.

(d) Determine the value of the integral I if C is the circle $|z| = 1/2$.

1. A point bead of mass m slides without friction along a circular wire of radius r and mass M. The plane of the wire hoop is exactly vertical and remains so during the motion. The hoop rolls in the x-direction without slipping on friction along a horizontal plane with the bottom of the hoop being always in contact with the x-axis. Gravity acts downwards in y direction. At time t = 0 the hoop is at rest at x = 0 and the bead is at the top of the hoop with velocity v_0 in the positive x direction.
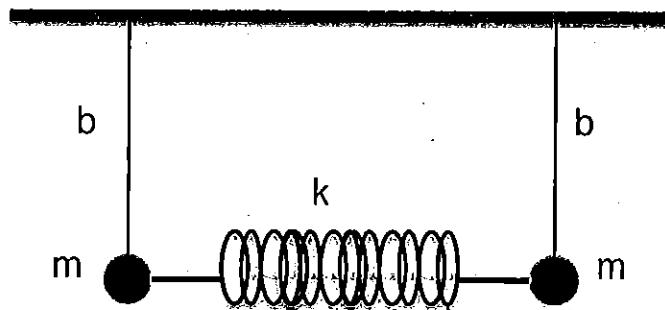
(a)   Using the constraints to determine appropriate generalized coordinates, obtain the Lagrangian and determine the equations of motion of the system.

(b)   Identify all conserved quantities in the motion of the system arising from cyclic coordinates.

2. Two pendula of equal length (b) having equal masses (m) are connected by a spring (spring constant k) as shown in the figure. Consider only small oscillations, and show that the eigenfrequencies are
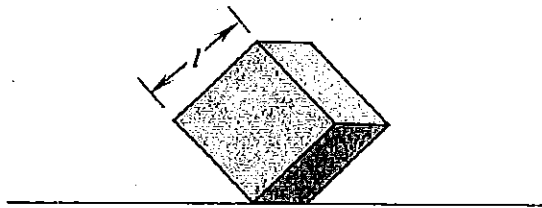
$$\omega = \sqrt{\frac{g}{b}} \qquad \text{and} \qquad \omega = \sqrt{\frac{g}{b} + \frac{2k}{m}} .$$

Find and describe the normal modes and coordinates by identifying the symmetric and anti-symmetric modes.
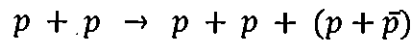
3. A homogeneous cube of side l is initially at rest in unstable equilibrium with one edge in contact with a horizontal plane. The cube is given a small angular displacement and allowed to fall. What is the angular velocity of the cube when one face contacts the plane if:

(a) the edge in contact with the plane cannot slide?

(b) the plane is frictionless so the edge can slide?

4. Consider an particle accelerator consisting of a beam of protons that collide with protons stationary in the lab frame. This accelerator produces antiprotons, $\bar{p}$, by the reaction

$$p + p \rightarrow p + p + (p + \bar{p})$$
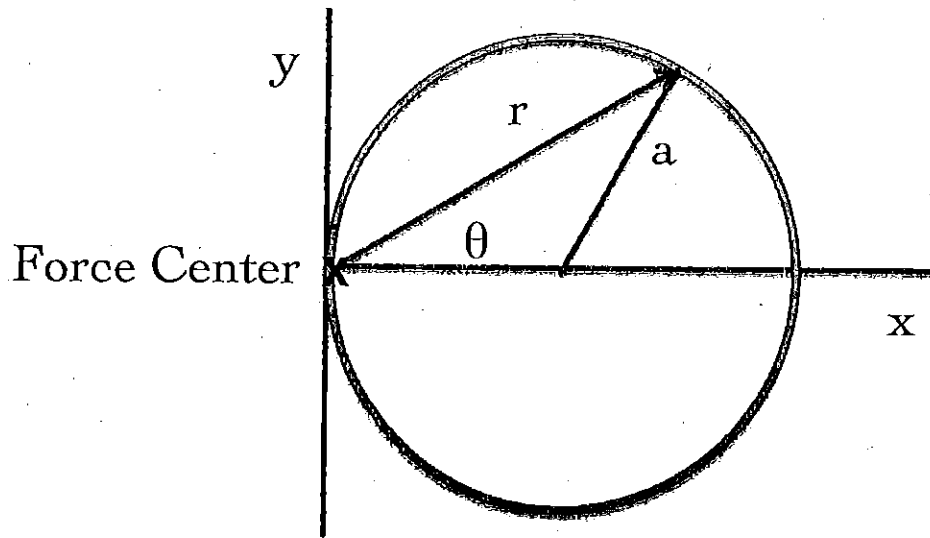
What is the minimum kinetic energy for each particle to produce this reaction...

a) in the center of mass frame?
b) in the lab frame?

[Note: The rest energy for protons and antiprotons is the same and equal to 938 MeV.]

5. A particle moves under the influence of a central force, given by
$F(r) = -k/r^n$, k = constant.

If the particle's orbit is a circle, passing <u>through</u> the force center, show that n=5

Recall that for central forces:

$$\frac{d^2}{d\theta^2}\left[\frac{1}{r(\theta)}\right] + \frac{1}{r(\theta)} = -\frac{mr^2}{l^2} F(r)$$

6. Consider a world where the Lagrangian of a particle is given by

$$L = \frac{\lambda v^4}{2} - \kappa \cosh(ax)$$

a) What are the units of λ and κ?

b) How do you know that energy is conserved?

c) Find the expression for the energy.

      [Hint: use the Hamilton function to express the energy.]

d) Consider the initial conditions:

$$x(t = 0) = x_0$$
$$v(t = 0) = 0.$$
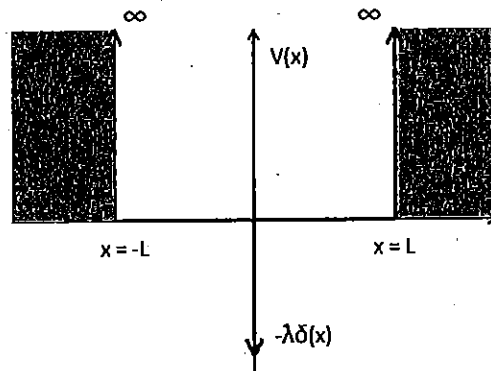
Without solving any differential equation, calculate,

$$v_f = v(x=0).$$

Express your answer for v_f in terms of the Lagrangian parameters (λ and κ), and the initial conditions.

1. Consider a particle with mass m bound to an attractive delta-function potential with its strength λ, *i.e.* $-\lambda\delta(x)$, positioned at the center of an infinite square well of width $2L$:

$$V(x) = \begin{cases} \infty, & x < -L \\ 0, & -L < x < 0 \\ -\lambda\delta(x), & x = 0 \\ 0, & 0 < x < L \\ \infty & x > L \end{cases} \tag{1}$$

which allows a state with $E < 0$.

(a) You can express the equation of the eigenstate by:

$$\psi(x) = \begin{cases} \psi_+(x) = Ae^{kx} + Be^{-kx}, & 0 < x < L \\ \psi_-(x) = Ce^{kx} + De^{-kx}, & -L < x < 0 \end{cases} \tag{2}$$

How does the coefficient k relates to the energy of the state E?

(b) Prove that the equation determining the energy E of the eigenstate can be given by

$$\tanh(kL) = \frac{k\hbar^2}{m\lambda} \tag{3}$$

2. The Hamiltonian operator of a two-state system takes the form,

$$H = E\Big(|+\rangle \langle +| + |+\rangle \langle -| + |-\rangle \langle +| - |-\rangle \langle -| \Big),$$

in the basis $\big(|+\rangle , |-\rangle \big)$, where E is a constant.

(a) Find the eigenvalues of H.

(b) Find the eigenkets of H in terms of $|+\rangle$ and $|-\rangle$.

3. A spin-$\frac{1}{2}$ particle moves within a spherically symmetry potential $V(r)$. Its total angular momentum **J** is given by $\mathbf{J} = \mathbf{L} + \mathbf{S}$.

(a) Since $\mathbf{L}^2$, $\mathbf{S}^2$, L_z and S_z all commute, one can choose $|ls; m_l m_s\rangle$ as the eigenstates of the particle, where

$$\mathbf{L}^2 |m_l, m_s\rangle = l(l+1)\hbar^2 |m_l, m_s\rangle, \quad L_z |m_l, m_s\rangle = m_l \hbar |m_l, m_s\rangle,$$

$$\mathbf{S}^2 |m_l, m_s\rangle = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 |m_l, m_s\rangle, \quad S_z |m_l, m_s\rangle = m_s \hbar |m_l, m_s\rangle,$$

with $|m_l, m_s\rangle \equiv |ls; m_l m_s\rangle$ and $m_s = \pm\frac{1}{2}$. For $l = 1$, write dow all the possible eigenstates $|m_l, m_s\rangle$.

(b) Since $\mathbf{L}^2$, $\mathbf{S}^2$, $\mathbf{J}^2$ and J_z commute, once can also choose $|ls; jm\rangle$ as the eigenstates of the particle, where

$$\mathbf{J}^2 |jm\rangle = j(j+1)\hbar^2 |jm\rangle, \quad J_z |jm\rangle = m\hbar |jm\rangle,$$

$$\mathbf{L}^2 |jm\rangle = l(l+1)\hbar^2 |jm\rangle, \quad \mathbf{S}^2 |jm\rangle = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 |jm\rangle,$$

with $|j, m\rangle \equiv |ls; jm\rangle$. For $l = 1$, write dow all the possible eigenstates $|jm\rangle$.

(c) Assume that $\left|\frac{3}{2}\frac{3}{2}\right\rangle \equiv \left|j = \frac{3}{2} m = \frac{3}{2}\right\rangle = \left|m_l = 1, m_s = \frac{1}{2}\right\rangle \equiv |1, +\rangle$, using the operators $J_\pm \equiv J_x \pm i J_y$, find all the eigenstates $\left|\frac{3}{2}m\right\rangle$ in terms of the eigenstates $|m_l, m_s\rangle$, where and

$$J_\pm |jm\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\,\hbar |jm \pm 1\rangle.$$

4. Using the trial function $\psi(x) = xe^{-\alpha x}$, find the best bound on the ground state energy of a quantum mechanical particle of mass m in the potential:

$$V(x) = \begin{cases} \infty & \text{for } x < 0 \\ cx & \text{for } x > 0 \end{cases}.$$

Simplify your answer as much as possible. [The exact value (to six significant figures) is $1.85576 \ \hbar^{2/3} m^{-1/3} c^{2/3}$]

5. The Wigner-Eckart Theorem states that the matrix element of a spherical tensor angular momentum operator can be factored into a part that depends on the orientation and a part that does not,

$$\langle j'm' \mid T_q^k \mid jm \rangle = \frac{\langle (jk)mq \mid (jk)j'm' \rangle}{\sqrt{2j'+1}} (j' \parallel T^k \parallel j),$$

where $\langle (jk)mq \mid (jk)j'm' \rangle$ is a Clebsh-Gordan coefficient and $(j' \parallel T^k \parallel j)$ (the orientation-independent factor) is called a reduced matrix element.

Consider a two-particle state $\mid Sm_s \rangle$ describing two spin-1/2 particles $\mid \frac{1}{2}m_1 \rangle$ and $\mid \frac{1}{2}m_2 \rangle$. The spin operator for this state is $S = \frac{\sigma_1 + \sigma_2}{2}$ where σ are the Pauli matricies, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The spherical tensor components of the (rank-one) S are given by $S_\pm = \mp \frac{1}{\sqrt{2}}(S_x \pm iS_y)$ and $S_0 = S_z$.

Show that $\langle 1 \parallel S \parallel 1 \rangle = \sqrt{6}$. You may find the following abbreviated table of Clebsch-Gordan coefficients useful:
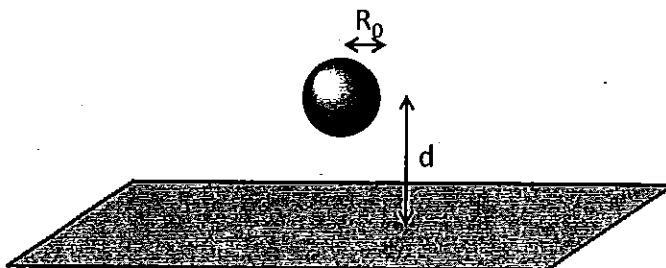
$$\langle (11)0{-}1 \mid (11)1{-}1 \rangle = \frac{1}{\sqrt{2}} \qquad \langle (11)1{-}1 \mid (11)10 \rangle = \frac{1}{\sqrt{2}} \qquad \langle (11)10 \mid (11)11 \rangle = \frac{1}{\sqrt{2}}$$

$$\langle (11){-}10 \mid (11)1{-}1 \rangle = -\frac{1}{\sqrt{2}} \qquad \langle (11)00 \mid (11)10 \rangle = 0 \qquad \langle (11)01 \mid (11)11 \rangle = -\frac{1}{\sqrt{2}}$$

$$\langle (11){-}11 \mid (11)10 \rangle = -\frac{1}{\sqrt{2}}$$

6. Consider the one-dimensional system of a partcile of rest mass m moving in the potential

$$V(x) = V_0 \alpha |x|,$$

where $-\infty < x < \infty$, $\alpha > 0$, $V_0 > 0$. Use WKB methods to estimate the bound state energies E_n, $n = 0, 1, 2, \ldots$, for the particle.

1. A uniformly-charged solid sphere of radius R_0 with charge density ρ_0 exists above a uniformly-charged sheet with surface charge density σ_1. The center of the sphere is away from the sheet by a distance d as shown below.

(a) What is the electric field due to the sphere?
(b) What is the electric field due to the uniformly-charged sheet?
(c) What is the potential difference between the center of the sphere and the nearest point on the sheet?
(d) What is the net electrostatic force on the sphere?

2. a) Using Biot-Savart law, find the magnetic field a distance z above the center of a circular loop of radius a, which carries a steady current I.

   b) Use the result from part a, calculate the magnetic field at the center of a uniformly charged spherical shell, of radius R and total charge Q, spinning at constant angular velocity ω.

3. From Maxwell's equations in free space with a given charge density   and current density **J**, show that

a) the **E** and **B** fields obey the wave equation,

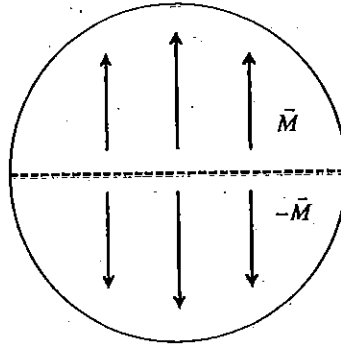b) the electric charge is a conserved quantity.

Figure 1: Exercise 4 magnet.

4. (Gaussian units) A spherical magnet of radius a has a split magnetic profile. The top half of the magnet has a uniform magnetization $\vec{M}$ in the positive z-direction, and the bottom half has a magnetization in the $-\vec{M}$ direction. The magnetic scalar potential is given by

$$\Phi(\vec{x}) = \oint_{S} da' \frac{\vec{M}(\vec{x}') \cdot \hat{n}'}{|\vec{x} - \vec{x}'|}.$$

(a) Find the magnetic field $\vec{B}$ far away from the magnet ($r \gg a$). (5 pts.)

(b) Find a form for the magnetic scalar potential as an expansion over Legendre polynomials $P_\ell(x)$. You may use the following ingredients. The Coulomb expansion,

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{\ell,m} \frac{r_<^\ell}{r_>^{\ell+1}} \frac{4\pi}{2\ell+1} Y_{\ell m}(\theta, \phi) Y_{\ell m}^*(\theta', \phi'),$$

where the $Y_{\ell m}(\theta, \phi)$ are the spherical harmonics. The connection between Legendre polynomials and spherical harmonics,

$$Y_{\ell 0}(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta).$$

The Legendre polynomial equation* as an expansion

$$P_\ell(x) = \frac{1}{2^\ell} \sum_{r=0}^{[\ell/2]} \frac{(-1)^r (2\ell - 2r)!}{r!(\ell - r)!(\ell - 2r)!} x^{\ell - 2r}.$$

Note that the upper limit $[\ell/2]$ in the r-sum denotes the greatest integer contained in the quantity $\ell/2$. (5 pts.)

5. The relation connecting charges and voltages on a collection of N conductors is

$$Q_i = \sum_{j=1}^{N} C_{ij} V_j,$$

where the C_{ij} are the coefficients of capacitance.

(a) (14 pts.) Find the 4 coefficients of capacitance, C_{aa}, C_{ba}, C_{ab}, and C_{bb} for two hollow concentric conducting spheres of radii b and a ($b > a$).

(b) (6 pts.) Given that the outer sphere of radius b is grounded and the inner sphere of radius a is given a potential V_a, find the amount of charge on the inner sphere.

6. (Gaussian units) An arbitrary charge density $\rho(\vec{x})$ exists outside of a spherical perfect electric conductor of radius a. Show that the total field outside the conductor can be written as the direct field due to $\rho(\vec{x})$ and an image charge $\rho^*(\vec{x})$ located inside the sphere, where the image charge is related to the direct charge in spherical coordinates centerd onthe sphere's center by

$$\rho^*(r, \theta, \phi) = -\left(\frac{a}{r}\right)^5 \rho(\frac{a^2}{r}, \theta, \phi).$$

or

$$\rho^*(\frac{a^2}{r}, \theta, \phi) = -\left(\frac{r}{a}\right)^5 \rho(r, \theta, \phi).$$

Note that the combination should give an $\vec{E}$ field which satisfies the boundary condition,

$$\vec{E} \times \hat{n}|_{r=a} = 0.$$

1. The Joule-Thompson coefficient is the coefficient of change in temperature with pressure at constant enthalpy $\left(\frac{\partial T}{\partial p}\right)\Big|_{H}$ and is usually given in the form

$$\left(\frac{\partial T}{\partial p}\right)\Big|_{H} = \frac{V}{C_p}\left(\alpha_b T - 1\right),$$

where T is temperature, V is volume, C_p is the heat capacity and

$$\alpha_b = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$$

is the bulk expansion coefficient.

(a) (4 pts.) Given the ideal gas equation of state,

$$p = \frac{N}{V}kT,$$

show that

$$\left(\frac{\partial T}{\partial p}\right)\Big|_{H} = 0.$$

(b) (16 pts.) Given the equation of state for an imperfect gas,

$$p = \frac{N}{V}kT\left(1 + \frac{N}{V}B_2\right),$$

where B_2 is the "virial coefficient", show that to first order in this coefficient that

$$\left(\frac{\partial T}{\partial p}\right)\Big|_{H} \approx \frac{N}{C_p}\left(T\left(\frac{\partial B_2}{\partial T}\right)_p - B_2\right).$$

2. The Bose - Einstein distribution is given by

$$n(\epsilon) = \frac{g(\epsilon)}{e^{\alpha + \frac{\epsilon}{kT}} - 1}$$

where n(ϵ) is the number of particles with energy ϵ, the density of energy states is g(ϵ) dϵ, k is Boltzmann's constant, and T is temperature $\left(g(p)dp = \frac{2}{h^3} 4\pi p^2 dp \right)$, where p is the photon momentum.

a) Given the fact that $\alpha = 0$ for a photon gas, calculate the energy density of photons of frequency ν in the range dν in thermal equilibrium.

b) What is the total energy density of the photon field?

c) The pressure of a perfect gas is $P = \frac{1}{3} \int p v_p n(p) dp$. Find the pressure of a photon gas. How is it related to the total energy density of the gas?

3. Consider a surface on which particles can be adsorbed and localized. Each site can accommodate 0, 1, or 2 particles. We neglect any interaction between particles even if they are localized on the same site. So for each number of particles adsorbed, the energy of the site is 0, -ε, or -2ε, respectively.

(a) Calculate the grand canonical partition functions for 1 site and then N sites.

(b) Using the grand canonical partition function, derive the mean number of adsorbed particles per site $<n>$ and the mean internal energy per site $<E>$ as a function of temperature T, chemical potential μ, and one particle adsorption energy ε.

4. Suppose that the measured temperature of the air above the arctic permafrost is expressed as a Fourier series

$$T(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos n\omega t,$$

where $2\pi/\omega$ is one year. Solve the heat equation

$$\frac{\partial T(z,t)}{\partial t} = \kappa \frac{\partial^2 T(z,t)}{\partial z^2}, \quad 0 < z < \infty$$

to find the soil temperature $T(z,t)$ at a depth z below the surface as a function of time t. Note that at the surface ($z = 0$), the soil temperature must match the air temperature given above.

You should observe that the sub-surface temperature fluctuates with the same period as that of the air, but with a phase lag that depends on the depth.

(Hint: Write the solution as a Fourier series with components $\mathrm{Re}\left[A_n(z)e^{in\omega t}\right]$, where A_n is a complex function of z.)

5. Prove the following result by residue theorem,

$$I = \int_0^\infty \frac{x \sin x\, dx}{x^2 + 1} = \frac{\pi}{2e}.$$

6. Consider the following equation

$$mX''(t) + bX'(t) + kX(t) = F(t), \quad t \in (0, +\infty),$$
$$X(0) = 0, \quad X'(0) = a \ ,$$

a damped oscillator in driven by a force $F(t)$. Find the displacement as a function of time by Laplace transform method, where m is the mass on a spring and k is the spring constant .
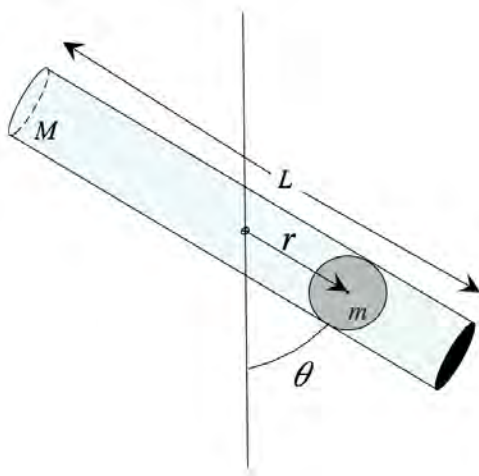
Baylor University Physics Ph.D. Program
Preliminary Exam 2014

Part I: Classical Mechanics
Friday, August 15, 2014, 9:00 am

There are six problems, each worth 20 points. After all six problems are graded, the top five scores will be totaled. Maximum points: 100

Problem 1

A particle of mass m is placed in a smooth, uniform tube of mass M and length L, as shown in the figure below. The tube is free to rotate about its center in a vertical plane. The system is started from rest with the tube horizontal and the particle is at the center of the tube. Take the moment of inertia of the tube to be $ML^2/12$.

(a) Find the Lagrangian and the Euler-Lagrange equations of motion for this system in terms of r and θ.
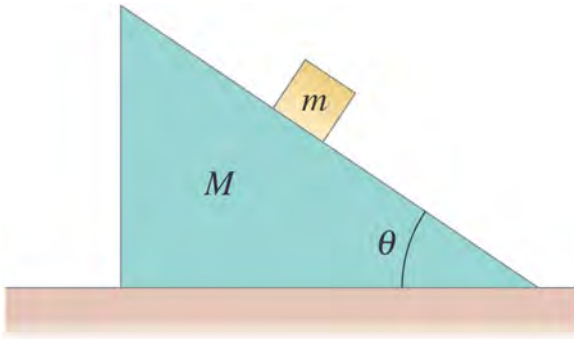
Sometime after the tube is released from rest and it begins to move, the angular velocity of the tube $\dot{\theta}$ will reach its maximum value. Let this maximum value be denoted by ω. When $\dot{\theta} = \omega$, the angle θ is equal to θ_m.

(b) When $\theta = \theta_m$, find the length of the tube L for which the particle will leave the tube. Your answer should be in terms of m, M, g, ω, and θ_m.

(Hint: In addition to the results from part (a), consider conservation of energy. At some point in your solution, you might need to solve a quadratic equation.)

A block of mass m can slide on a wedge of mass M which, in turn, can slide on a horizontal surface as shown in the figure.

In the steps below, you will find the acceleration of the block on the wedge, the acceleration of the wedge, and the interaction force between the block and the wedge. Assume all surfaces are frictionless surfaces.
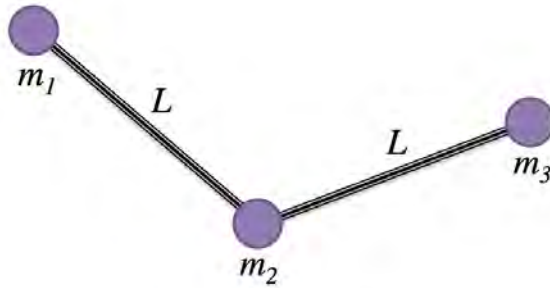
(a)   Choose a proper set of generalized coordinates and a constraint function to solve the problem.

(b)   Obtain the kinetic energy, the potential energy, and the Lagrangian for the system using your generalized coordinates.

(c)   Determine the equations of motion for the system.

(d)   Solve them to find the acceleration of the block and the acceleration of the wedge. Verify that the acceleration of the block is reasonable for the case where $\theta = 90°$.

(e)   Find the interaction force between the block and the wedge and show that it can be expressed as
$$F = \frac{Mmg}{(M+m)\sec\theta - m\cos\theta}.$$

(f)   Identify the conserved quantities that arise from the equations of motion and explain why they are conserved.

Three hard spheres with masses m_1, m_2, and m_3 in the ratio $m_1 : m_2 : m_3 = 1:2:1$ are connected by two light, flexible (stretchable and bendable) rods, both of length L. The masses of the rods are negligible.

(a) Assuming only small displacements in the plane formed by the spheres, determine all the normal modes of the system (including when the motion of all three spheres is in a straight line and when it is not).

(b) State what you can about the relative frequencies of the normal modes.

Problem 4

Muons created by cosmic rays move at highly relativistic speeds in the earth's atmosphere.

The mean lifetime of muons is 2.197 μs in their own rest frame. If exactly 10,000 muons are produced by cosmic rays 40.26 km above the earth's surface, and they travel directly toward the earth's surface with a speed of $0.9940\,c$, how many of these muons are predicted to reach the earth's surface?

Use $c = 2.998 \times 10^8$ m/s for the speed of light.

Problem 5

An object is projected horizontally with speed v_o from a point located at height h above ground level at a latitude λ in the southern hemisphere. Assume $h \ll R_E$, where R_E is the radius of the earth.

(a)   Assuming the z axis is in the vertical direction, obtain the equations of motion in all three directions to study the deflection of the object due to the Coriolis effect.

(b)   If the object is projected toward the east, modify your answer in part (a) to study the deflection (making a first-order approximation in ω).

(c)   Find the deflection of the object when it lands on the ground (once again, to first order in ω). Give your answer in the simplest form.

(d)   What is the direction of deflection? Your answer must be consistent with your result in part (c).

Problem 6

A very small hole of radius R is cut from an infinite flat sheet with mass density σ (mass per unit area). Assume the sheet is in a remote location in space with no other nearby masses. Let L be the line that is perpendicular to the sheet and passes through the center of the hole.

(a) Find the gravitational force on a point mass m that is located on L, at a distance x from the center of the hole. Assume that x and R are approximately the same size.

(b) If a particle is released from rest on L, very close to the center of the hole ($x \ll R$), show that it undergoes oscillatory motion, and find the frequency of small oscillations.

(c) If a particle is released from rest on L, at a distance x from the sheet ($x > R$), what is its speed when it passes through the center of the hole?

Part II: Quantum Mechanics
Friday, August 15, 2014, 1:30 pm

There are six problems, each worth 20 points. After all six problems are graded, the top five scores will be totaled. Maximum points: 100

Problem 1

A particle of mass m is in a harmonic oscillator potential with angular frequency ω. The initial state of the particle is a linear combination of $|0\rangle$ and $|1\rangle$, where the particle is three times more likely to be in state $|0\rangle$ than in state $|1\rangle$.

(a) Calculate the properly normalized initial state $|\psi(0)\rangle$.

(b) Calculate

(1) $\Delta x = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2}$

(2) $\Delta p = \sqrt{\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2}$

(3) $\Delta x \, \Delta p$

and comment on whether or not the product (3) obeys Heisenberg's uncertainty relation.

(c) Show that momentum is conserved for the particle in this initial state.

Consider an atom with two valence electrons with spins $\vec{s}_i$ and total angular momenta $\vec{j}_i$ where $i = 1$ or 2, in the usual notation. The atom is in such a state that, in the j-j coupling scheme, each valence electron has $j = 3/2$ for its total angular momentum eigenvalue while the total angular momentum $\vec{J} = \vec{j}_1 + \vec{j}_2$ of the two valence electrons is $\vec{0}$. Thus, the atom is in the state $|3/2, 3/2; 0\rangle$ in the j-j coupling scheme.

Find the representation of this state in the LS-coupling scheme in terms of the states $\{|^{2S+1}L_J\rangle\}$, where L is the total orbital angular momentum eigenvalue, S is the total spin angular momentum eigenvalue, and J is the total angular momentum eigenvalue.

Problem 3

Consider an electron in a hydrogen atom and ignore relativistic effects. The Schrödinger equation is written

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(\vec{r}) - \frac{e^2}{4\pi\epsilon_0 r}\psi(\vec{r}) = E\psi(\vec{r}).$$

in spherical polar coordinates.

Suppose the electron were in a stationary state with spatial wave function

$$\psi(\vec{r}) = Cre^{-r/(2a_0)}\cos\theta$$

written in spherical polar coordinates, where C is a constant and

$$a_0 = 4\pi\epsilon_0\frac{\hbar^2}{me^2}$$

is the Bohr radius. A series of experiments is now performed.

(a) If the energy were measured, what would it be?

(b) If the magnitude of the angular momentum were measured, what would it be?

(c) If the z-component of the angular momentum were measured, what would it be?

(d) If the x-component of the angular momentum were measured, what would it be?

Problem 4

Consider the *double* delta-function potential

$$V(x) = -\beta \left[\delta(x+a) + \delta(x-a)\right],$$

where β and a are positive constants.

(a) Sketch the potential.

(b) By applying appropriate boundary conditions, one can find the following transcendental equation for the bound-state energies corresponding to states that are *even*, namely $\psi(-x) = \psi(x)$:

$$z = \gamma(1 + e^{-2z}) \quad \text{where} \quad z \equiv ka, \quad \gamma \equiv \frac{ma\beta}{\hbar^2}, \quad \text{and} \quad k^2 = -\frac{2mE}{\hbar^2}.$$

Derive a similar transcendental equation for the bound-state energies corresponding to states that are *odd*, namely $\psi(-x) = -\psi(x)$.

(c) Determine how many bound states (even and/or odd) this potential possesses for the special cases:

(1) $\beta = \dfrac{\hbar^2}{3ma}$

(2) $\beta = \dfrac{2\hbar^2}{ma}$

Problem 5

A particle is in a one-dimensional potential

$$
V(x) = \begin{cases} 0 & \text{for} \quad 0 < x < a \\ V_0 & \text{for} \quad a < x < a+b \\ \infty & \text{for} \quad x < 0 \quad \text{or} \quad x > a+b \end{cases}
$$

where $V_0 > 0$.

(a) Assuming the total energy E of the particle is greater than V_0 ($E > V_0$), find the constraints on the discrete momentum levels of the particles using the Wilson-Sommerfeld momentum quantization condition

$$
\int p \, dx = n \, h, \qquad n = 1, 2, \ldots
$$

where $\int$ implies a complete cycle in the x direction.

(b) Assuming the total energy E of the particle is greater than V_0 ($E > V_0$), find the constraints on the discrete momentum levels of the particles by solving the Schrödinger equation in the regions $0 < x < a$ and $a < x < b$ and applying continuity across the boundary at $x = a$.

(c) Compare the two constraints. Are the constraints always equal only in a specific limit? If only in a specific limit, what is this limit and why?

Problem 6

Consider the operator

$$
L_y = \begin{pmatrix} 0 & \dfrac{-i}{\sqrt{2}} & 0 \\[2mm] \dfrac{i}{\sqrt{2}} & 0 & \dfrac{-i}{\sqrt{2}} \\[2mm] 0 & \dfrac{i}{\sqrt{2}} & 0 \end{pmatrix}
$$

and the initial state

$$
|\psi\rangle = \begin{pmatrix} \dfrac{1}{\sqrt{3}} \\[2mm] \dfrac{1}{\sqrt{3}} \\[2mm] -\dfrac{1}{\sqrt{3}} \end{pmatrix}.
$$

(a) Find $\langle L_y \rangle$.

(b) Find ΔL_y.

(c) Find the eigenvalues and normalized eigenvectors of operator L_y.

(d) If L_y operates on the state $|\psi\rangle$, what is the probability that the outcome will be $+1$?

(e) What is the probability that the outcome from part (d) will be $+\frac{1}{2}$?

Baylor University Physics Ph.D. Program
Preliminary Exam 2014

Part III: Electricity & Magnetism
Saturday, August 16, 2014, 9:00 am

There are six problems, each worth 20 points. After all six problems are graded, the top five scores will be totaled. Maximum points: 100

Problem 1

An non-conducting circular ring of radius a lies in the xy plane, centered at the origin. It carries a linear charge density $\lambda = \lambda_0 \sin \frac{\phi}{2}$, where λ_0 is a constant and ϕ is the azimuthal angle. You spin the ring about its axis at an angular velocity ω in the counterclockwise $(+\phi)$ direction. (Hint: For this problem, you may find it useful to consider retarded potentials.)

(a) Find the scalar potential $V(t)$ at the center of the ring.

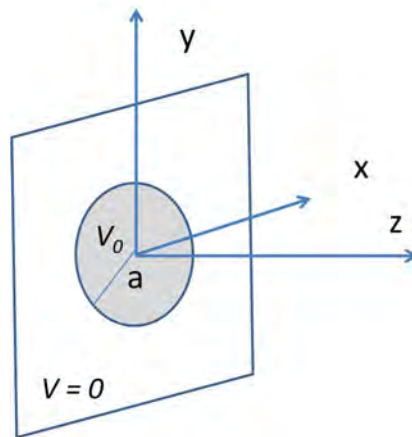**(b)** Find the vector potential $\vec{A}(t)$ and show that it can be written as

$$\vec{A}(t) = \frac{\mu_0 \lambda_0 \omega a}{3\pi} \left\{ \sin\left[\omega(t - a/c)\right] \hat{\mathbf{i}} - \cos\left[\omega(t - a/c)\right] \hat{\mathbf{j}} \right\}.$$

(c) Find the dipole moment $\vec{p}(t)$ of the ring.

Problem 2

A large conducting plane is in the xy plane with its center at the origin. It is maintained at a potential V_0 within a circular disc of radius a and grounded elsewhere. The potential $V(\rho, \phi)$ on the plane is
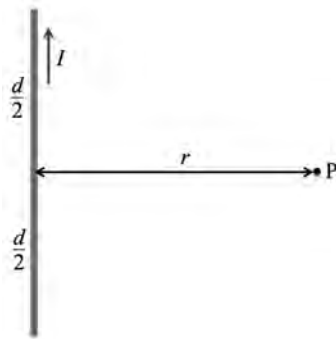
$$V(\rho, \phi) = \begin{cases} V_0, & \rho \le a \\ 0, & \rho > a \end{cases}$$

(a) What is the Green's function for this problem?

(b) What is the potential Φ on the z axis?

Problem 3

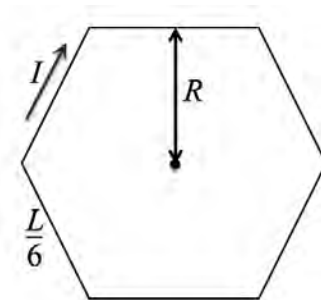**(a)** A straight section of wire of length d carries a current I as shown in the figure.

Use the Biot-Savart law to show that the magnitude of the magnetic field at a point P a distance r from the wire along the perpendicular bisector is

$$B = \frac{\mu_0 I}{2\pi r} \frac{d}{\sqrt{d^2 + \alpha\, r^2}}$$

in SI units, where α is some constant. Determine the value of α.

(b) A wire of length L, carrying current I, is bent into the shape of a regular polygon with n sides whose sides are a distance R from the center. (The figure below shows the special case of $n = 6$.)
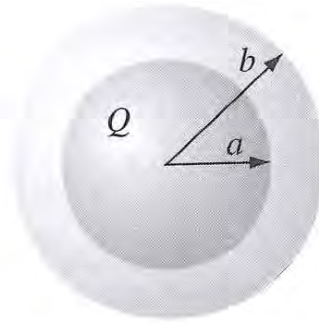

Use the result of part (a) to help you find the magnitude of the magnetic field at the center of the polygon as a function of μ_0, I, n, and L.

(c) Show how your answer to part (b) gives the expected result for the magnitude of the magnetic field at the center of a circular wire loop of radius R (circumference L) as $n \to \infty$.

3

Problem 4

A spherical conductor of radius a carries a charge Q as shown below. The conductor is surrounded by linear dielectric material of susceptibility χ_e out to radius b.

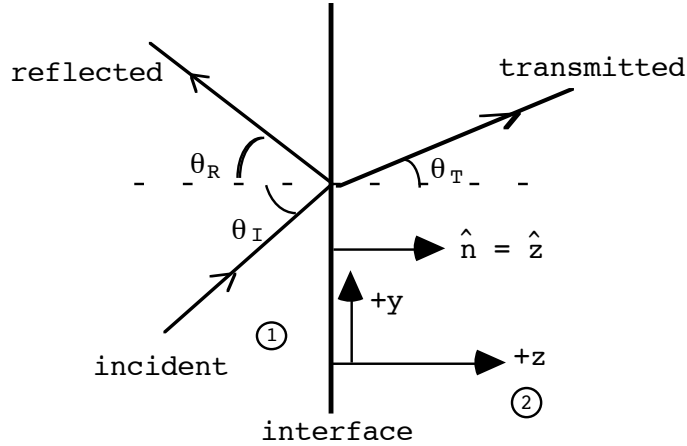Hint: $\vec{\mathbf{P}} = \epsilon_0 \chi_e \vec{\mathbf{E}}$ and $\epsilon = \epsilon_0 (1 + \chi_e)$.

(a) Find the electric field $\vec{\mathbf{E}}$, the electric displacement $\vec{\mathbf{D}}$, and the polarization $\vec{\mathbf{P}}$

    (1) inside the spherical conductor,

    (2) within the dielectric material, and

    (3) outside the dielectric material.

(b) Find the bound charge ρ_b within the dielectric material, and the bound charge σ_b on both the inner and outer surfaces of the dielectric material.

(c) Find the energy of the configuration.

Consider the scattering of plane waves from a two-dimensional interface, as shown in the figure below.

We define

$$f_1(\vec{\mathbf{x}}, t) \equiv f_I(\vec{\mathbf{x}}, t) + f_R(\vec{\mathbf{x}}, t) \quad (z < 0)$$
$$f_2(\vec{\mathbf{x}}, t) \equiv f_T(\vec{\mathbf{x}}, t) \qquad\qquad (z > 0)$$

where $\vec{\mathbf{x}} = (y, z)$ and

$$f_I(\vec{\mathbf{x}}, t) = A_I \exp\left[i(\vec{\mathbf{k}_I} \cdot \vec{\mathbf{x}} - \omega t)\right]$$

$$f_R(\vec{\mathbf{x}}, t) = A_R \exp\left[i(\vec{\mathbf{k}_R} \cdot \vec{\mathbf{x}} - \omega t)\right]$$

$$f_T(\vec{\mathbf{x}}, t) = A_T \exp\left[i(\vec{\mathbf{k}_T} \cdot \vec{\mathbf{x}} - \omega t)\right].$$

A_I, A_R, and A_T are the complex incident, reflection, and transmisison amplitudes, respectively, and $\vec{\mathbf{k}_I}$, $\vec{\mathbf{k}_R}$, and $\vec{\mathbf{k}_T}$ are the corresponding wave numbers. By definition, the z components of $\vec{\mathbf{k}_R}$ and $\vec{\mathbf{k}_I}$ are equal in magnitude but opposite in sign: $(\vec{\mathbf{k}_R})_z = -(\vec{\mathbf{k}_I})_z$.

Take the boundary conditions **on the interface at $z = 0$** to be

$$\alpha_1 f_1 = \alpha_2 f_2 \qquad\qquad (i)$$
$$\beta_1\, \hat{\mathbf{n}} \cdot \vec{\nabla} f_1 = \beta_2\, \hat{\mathbf{n}} \cdot \vec{\nabla} f_2 \qquad (ii)$$

where $\hat{\mathbf{n}} = \hat{\mathbf{z}}$.

(a) Using the continuity condition (i), show that

$$\frac{\sin \theta_{\mathrm{T}}}{\sin \theta_{\mathrm{I}}} = \frac{k_{\mathrm{I}}}{k_{\mathrm{T}}},$$

where k_{I}, k_{R}, and k_{T} are the magnitudes of the vectors $\vec{\mathbf{k}}_{\mathbf{I}}$, $\vec{\mathbf{k}}_{\mathbf{R}}$, and $\vec{\mathbf{k}}_{\mathbf{T}}$, respectively. This is a form of Snell's law.

(b) Using both boundary conditions (i) and (ii), show that

$$A_{\mathrm{T}} = \frac{2}{\dfrac{\alpha_2}{\alpha_1} + \gamma} A_{\mathrm{I}} \qquad \text{and} \qquad A_{\mathrm{R}} = \frac{\dfrac{\alpha_2}{\alpha_1} - \gamma}{\dfrac{\alpha_2}{\alpha_1} + \gamma} A_{\mathrm{I}}$$

where

$$\gamma = \frac{\beta_2(\vec{\mathbf{k}}_{\mathbf{T}})_z}{\beta_1(\vec{\mathbf{k}}_{\mathbf{I}})_z} = \frac{\beta_2 k_{\mathrm{T}} \cos \theta_{\mathrm{T}}}{\beta_1 k_{\mathrm{I}} \cos \theta_{\mathrm{I}}}.$$

Problem 6

A sphere of radius a is filled uniformly with N circular current loops of radius b ($b \ll a$). Each current loop carries a current I, and the loops are oriented with their axes in the z direction.

(a) What is the magnetic dipole moment $\vec{m}$ of the sphere?

(b) What is the magnetization $\vec{M}$ as a function of position, both inside and outside the sphere?

(c) What are the magnetic fields $\vec{H}$ and $\vec{B}$ as a function of position, both inside and outside the sphere?

Baylor University Physics Ph.D. Program
Preliminary Exam 2014

Part IV: Statistical Mechanics & Mathematical Methods
Saturday, August 16, 2014, 1:30 pm

There are six problems, each worth 20 points. After all six problems are graded, the top five scores will be totaled. Maximum points: 100

Problem 1

Let us model the universe as a spherical cavity with radius $R = 10^{26}$ m and temperature $T = 3$ K.

(a) Find the total number of thermally excited photons in the universe.

(b) Find the total energy of these photons.

(c) Using the results of parts (a) and (b), show that the average energy per photon is approximately 10^{-22} J.

(d) Provide an explanation for why the night sky is dark.

Hint: Along the way, you will demonstrate as a consequence that the energy in the universe is proportional to T^4.

Useful constants:

$$k_B = 1.38 \times 10^{-23} \text{ J/K}$$

$$h = 6.63 \times 10^{-34} \text{ J} \cdot \text{s}$$

Consider a solid in thermal equilibrium with its surroundings, at temperature T and pressure P. The solid is composed of N atoms and n vacancies, so there are a total of $N + n$ lattice sites. Take v to be the volume associated with each atom and vacancy.

The energy necessary to create each vacancy, through thermal excitation, is ϵ.

Use k_B for Boltzmann's constant.

(a) Find the entropy S in terms of the given quantities.

(b) Write down the the Gibbs free energy G in terms of the given quantities.

(c) Show that the creation of the vacancies in the solid causes the solid to expand.

Problem 3

You have a cup of very hot coffee and some cold milk. The coffee is initially too hot to drink. Which of the following methods would allow you to start drinking sooner?

(1) adding a spoonful of milk to cool the coffee slightly, then waiting for the coffee + milk mixture to reach a drinkable temperature, or

(2) waiting some amount of time for the coffee to cool to nearly a drinkable temperature, then adding a spoonful of milk?

You can assume that the temperature of the milk and the surroundings are at the zero of the temperature scale, and that coffee and milk have the same specific heat.

Use equations and fully justify your answer.

Problem 4

Consider the differential equation

$$\frac{d^2y}{dt^2} + k\frac{dy}{dt} = f(t),$$

with $y(0) = 0$, $y'(0) = 0$, and $f(t) = t$.

(a) Put the differential equation in Sturm-Liouville form.

(b) Find the Green's function corresponding to the Sturm-Liouville operator.

(c) Use your Green's function from part (b) to find $y(t)$.

(d) By direct substitution, show that your solution from part (c) satisfies the differential equation and the initial conditions.

Problem 5

A vector field $\vec{A}$ is given by

$$\vec{A}(x, y, z) = -\frac{xz}{r^3}\hat{i} - \frac{yz}{r^3}\hat{j} + \frac{x^2 + y^2}{r^3}\hat{k},$$

with $r^2 = x^2 + y^2 + z^2$.

(a) Establish that $\vec{A}$ is a conservative field by proving that $\vec{\nabla} \times \vec{A} = 0$.

(b) Find the potential function ϕ that corresponds to the vector field $\vec{A}$.

Problem 6

Consider a particle in a state described by

$$\psi(x, y, z) = N(2x + 2y + z)e^{-\alpha r^2},$$

where $r^2 = x^2 + y^2 + z^2$, α is a constant, and N is a normalization factor.

(a) Write this function in terms of spherical harmonics.

(b) Evaluate the normalization factor, N, to obtain a properly normalized wave function.

Baylor University Physics Ph.D. Program
Preliminary Exam 2015

Part I: Classical Mechanics
Friday, August 14, 2015, 9:00 am

There are six problems, each worth 20 points. After all six problems are graded, the top five scores will be totaled. Maximum points: 100

$\boxed{\text{Problem 1}}$

A cone of half-angle α stands on its tip, with its axis in the vertical direction. A small ring of radius r moves on the inside surface of the cone without slipping down. Assume that the conditions have been set so that (1) the point of contact between the ring and the cone moves in a circle at height h above the tip, and (2) the plane of the ring is at all times perpendicular to the line joining the point of contact and the tip of the cone. Assume that r is much smaller than the radius of circular motion R.
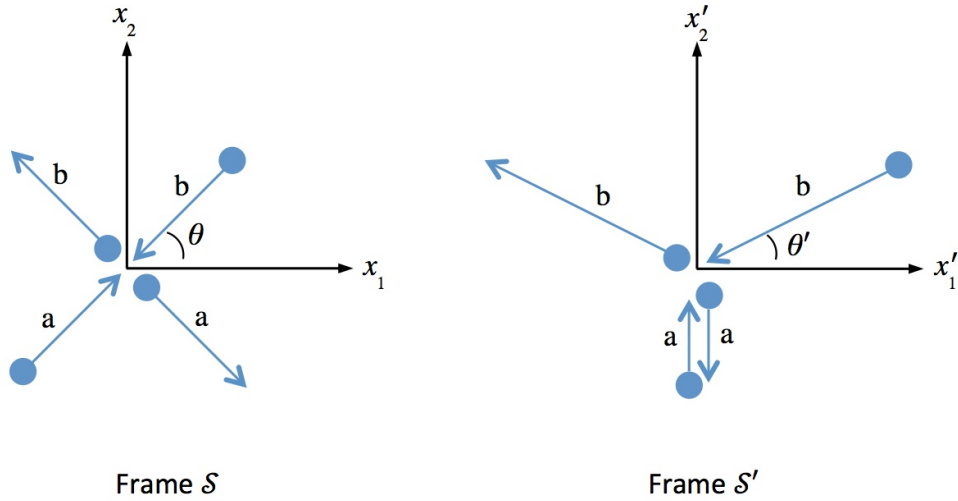

(a) Assuming that the surface of the cone is frictionless and the ring slides on the surface of the cone, find the angular speed Ω of the motion in terms of α, h, and g (the acceleration due to gravity).

(b) Assuming that the surface of the cone has friction and the ring rolls on it without sliding, find the angular speed Ω of the motion in terms of α, h, and g (the acceleration due to gravity).

Hint: For this problem, you might find it helpful to draw a free-body diagram for the ring.

Problem 2

In relativistic mechanics, the momentum of a body with mass m and velocity $\vec{v}$ is given by the four-vector $\vec{\mathbf{p}} = m\vec{\mathbf{u}} = (\gamma m\vec{\mathbf{v}}, \gamma mc)$. Consider an elastic collision between two particles with equal mass, labeled a and b, as shown in the figure below.

Frame $\mathcal{S}$ Frame $\mathcal{S}'$

In frame $\mathcal{S}$, the two particles approach with equal and opposite velocities (each with magnitude v) and emerge with their x_2 components reversed. Now consider the frame $\mathcal{S}'$, which is moving along the $+x_1$ axis of frame $\mathcal{S}$ with speed $v\cos\theta$. The speed $v\cos\theta$ is equal to the x_1 component of particle a's initial velocity in frame $\mathcal{S}$.

Using the Lorentz transformation that transforms the four-momenta of the particles from $\mathcal{S}$ to $\mathcal{S}'$, show that even though the x_2-component of the velocities of particle a and particle b are equal in magnitude and opposite in sign in $\mathcal{S}$, they are not in $\mathcal{S}'$:

$$(v_a)'_{2,\,\text{initial}} = +\frac{v\sin\theta}{\gamma(1 - \frac{v^2}{c^2}\cos^2\theta)} \qquad (v_a)'_{2,\,\text{final}} = -\frac{v\sin\theta}{\gamma(1 - \frac{v^2}{c^2}\cos^2\theta)}$$

$$(v_b)'_{2,\,\text{initial}} = -\frac{v\sin\theta}{\gamma(1 + \frac{v^2}{c^2}\cos^2\theta)} \qquad (v_b)'_{2,\,\text{final}} = +\frac{v\sin\theta}{\gamma(1 + \frac{v^2}{c^2}\cos^2\theta)}$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}\cos^2\theta}}.$$

Problem 3

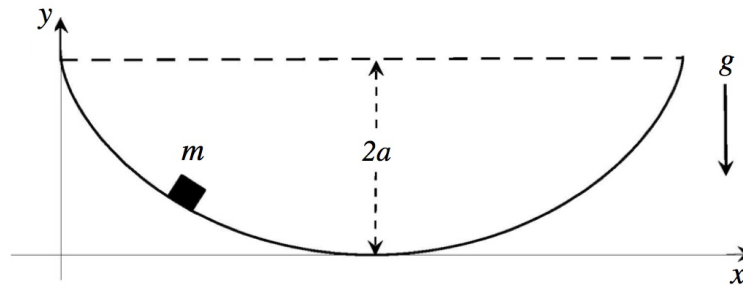Consider the following differential equations:

$$m\ddot{q}_1 + \epsilon\ddot{q}_2 + sq_1 = 0 \quad \text{and} \quad \epsilon\ddot{q}_1 + m\ddot{q}_2 + sq_2 = 0$$

with $s > 0$, $m > 0$, and $0 < \epsilon < m$.

(a) Solve for the squared eigenfrequencies, ω^2, of the motion.

(b) Find the time-independent ratio of the generalized coordinates q_1/q_2 for each normal mode.

A point mass m glides without friction on a cycloid given by

$$x = a(\theta - \sin\theta) \quad \text{and} \quad y = a(1 + \cos\theta) \qquad \text{for} \quad 0 \le \theta \le 2\pi.$$

The entire apparatus is in a uniform vertical gravitational field and the motion of the point mass m is in the x-y plane, as shown in the figure.

(a) Express the Lagrangian in terms of θ and $\dot{\theta}$.

(b) Determine the equation of motion.

(c) Let $u = \cos(\theta/2)$. Without assuming a small displacement of the point mass from the bottom of the cycloid, show that the exact solution for u satisfies the differential equation

$$\frac{d^2 u}{dt^2} + \frac{g}{4a} u = 0.$$

(d) Find the general solution for $u(t)$.

4

Problem 5

A particle of mass m moves along a trajectory given by

$$x(t) = x_0 \cos \omega_1 t \quad \text{and} \quad y(t) = y_0 \sin \omega_2 t.$$

(a) Find the x and y components of the force. Under what conditions is the force a central force?

(b) Find the potential energy V as a function of x and y.

(c) Determine the kinetic energy T of the particle. Show that the total energy E of the particle is conserved.

Note: Do not assume a central force for parts (b) and (c).

A sphere of radius a and mass m rests on top of a fixed sphere of radius b, as shown in the figure. The small sphere is slightly displaced so that it rolls without slipping down the large sphere. By completing the parts below, use the Lagrangian undetermined multiplier method to determine the point at which the small sphere leaves the large sphere.

(a) Find all constraints of motion and write the constraint functions.

(b) Determine the Lagrangian.

(c) Apply the Lagrangian undetermined multiplier method and find all the forces of constraint.

(d) Give the physical meaning of each force of constraint and determine the angle θ at which the small sphere leaves the large sphere.

Baylor University Physics Ph.D. Program
Preliminary Exam 2015

Part II: Quantum Mechanics
Friday, August 14, 2015, 1:30 pm

There are six problems, each worth 20 points. After all six problems are graded, the top five scores will be totaled. Maximum points: 100

Problem 1

Consider the harmonic oscillator Hamiltonian

$$H = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2.$$

Use WKB methods to estimate its energy eigenvalues.

Problem 2

The Hamiltonian of a system is given by

$$H = \hbar\omega \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(a) If the state of the system is described by

$$|\psi_1\rangle \rightarrow \frac{1}{10} \begin{pmatrix} 7 \\ \sqrt{2}i \\ 7 \end{pmatrix},$$

what are the possible results of an energy measurement and the corresponding probabilities?

(b) Given the state $|\psi_1\rangle$, you measure the observable

$$L_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

What is the probability of obtaining the value $+\hbar$? What is the state of the system immediately after the measurement? Denote this state by $|\psi_0\rangle$.

(c) Immediately after the measurement in (b), does the state of the system $|\psi_0\rangle$ have a well defined energy eigenvalue? If so, what is it? If not, explain why not.

(d) Given $|\psi_0\rangle$ from part (b), write an expression for $|\psi_t\rangle$, the state of the system at time t.

(e) Suppose we measure L_x on the system described by $|\psi_t\rangle$ from part (d). What is the probability of finding the value $+\hbar$? Specify the time at which the probability returns to the initial value.

(f) Compute the average value of L_x for the state $|\psi_t\rangle$ at time t.

A positively charged spin-½ particle with magnetic moment $\vec{\mu} = g\mu_B \vec{S}$ is at rest in the magnetic field $\vec{B} = B\,\hat{z}$ for time $t < 0$, and $\vec{B} = B\,\hat{x}$ for time $t > 0$.

(a) Write down the Hamiltonian of the system and find the energy eigenvalues and eigenstates of the system, both for $t < 0$ and for $t > 0$.

(b) If the particle is in its highest energy eigenstate for $t < 0$, what is its wave function for $t > 0$?

(c) For the wave function in part (b), find the expectation values of the components of $\vec{S}$ for all times t.

Problem 4

Assuming that $\psi_1(t, \vec{\mathbf{r}})$ and $\psi_2(t, \vec{\mathbf{r}})$ are two solutions of the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\psi(t, \vec{\mathbf{r}}) = -\frac{\hbar^2}{2m}\nabla^2\psi(t, \vec{\mathbf{r}}) + V(\vec{\mathbf{r}})\psi(t, \vec{\mathbf{r}}),$$

prove that $\int \psi_1^*\psi_2\, d^3x$ is independent of time t.

Problem 5

(a) Work out the following canonical commutation relations and show that

$$[r_i, p_j] = i\hbar\delta_{ij}$$

where the indices are x, y or z, and $r_x = x$, $r_y = y$, and $r_z = z$.

(b) Show that the time derivative of the expectation value of some observable $Q(x, p, t)$ can be expressed by:

$$\frac{d}{dt}\langle Q \rangle = \frac{i}{\hbar}\left\langle [\hat{H}, \hat{Q}] \right\rangle + \left\langle \frac{\partial Q}{\partial t} \right\rangle.$$

(c) Confirm Ehrenfest's theorem:

$$\frac{d}{dt}\langle \vec{\mathbf{r}} \rangle = \frac{1}{m}\langle \vec{\mathbf{p}} \rangle, \quad \text{and} \quad \frac{d}{dt}\langle \vec{\mathbf{p}} \rangle = \left\langle -\vec{\nabla}V \right\rangle.$$

Hint: You may consider using the outcome of part (b) to confirm this theorem.

Problem 6

(a) Consider a Hamiltonian $H(\lambda)$ that is dependent on some real parameter λ. Using the basic eigenvalue-eigenvector statement,

$$(H(\lambda) - E(\lambda))\,|E(\lambda)\rangle = 0$$

for a normalized eigenstate $|E(\lambda)\rangle$, prove the Feynman-Hellman theorem:

$$\langle E(\lambda)|\,\frac{\partial H}{\partial \lambda}\,|E(\lambda)\rangle = \frac{\partial E(\lambda)}{\partial \lambda}.$$

(b) The one-electron atom Hamiltonian is given in Gaussian units by

$$H = \frac{p^2}{2\mu} + V(r), \quad V(r) = -\frac{Ze^2}{r},$$

where Z is the number of protons in the nucleus, μ is the reduced mass, and r is the radial distance from the origin to the electron. The energy levels are

$$E_n = -\frac{Z^2 \mu e^4}{2\hbar^2 n^2},$$

where $n\,(=1, 2, 3, \dots)$ is the principal quantum number.

Use the result of (a) to show that the expectation value of the potential energy part of the Hamiltonian is given by

$$\langle V(r)\rangle = 2E_n.$$

There are six problems, each worth 20 points. After all six problems are graded, the top five scores will be totaled. Maximum points: 100

Problem 1

A square loop of wire (side a) lies on a table, a distance s from a very long straight wire, which carries a current I, as shown below.

(a) Find the flux of $\vec{B}$ through the loop.

(b) If someone now pulls the loop away from the wire at speed v, what electromotive force $\mathcal{E}$ is generated?

(c) In what direction (clockwise or counterclockwise) does the current flow?

(d) What happens if the loop is pulled to the *left* at speed v?

Problem 2

The retarded scalar potential V and retarded vector potential $\vec{A}$ in the radiation zone of an oscillating dipole, $\vec{p}(t) = p_0 \cos \omega t \, \hat{z}$, can be written as follows:

$$V(r, \theta, t) = -\frac{p_0 \omega}{4\pi\epsilon_0 c} \left(\frac{\cos\theta}{r}\right) \sin\left[\omega\left(t - \frac{r}{c}\right)\right]$$

$$\vec{A}(r, \theta, t) = -\frac{\mu_0 p_0 \omega}{4\pi r} \sin\left[\omega\left(t - \frac{r}{c}\right)\right] \hat{z}$$

(a) Determine the resulting electric field $\vec{E}$ and magnetic field $\vec{B}$ in the radiation zone.

(b) Find the intensity I of radiation in the radiation zone and show that

$$I = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\right) \frac{\sin^2\theta}{r^2}.$$

(c) Find the power radiated by this dipole over a spherical surface of radius r centered on the dipole.

(d) Do these potentials (V and $\vec{A}$) satisfy the Coulomb gauge? Do they satisfy the Lorentz gauge? Comment why or why not.

Problem 3

(a) Consider moving charges that give rise to a current density $\vec{\mathbf{J}}$ within a volume V in the presence of electric and magnetic fields. Show that the total power injected into the current distribution by the fields is given by

$$\int_V d^3x \ \vec{\mathbf{J}} \cdot \vec{\mathbf{E}}.$$

(b) Using Maxwell's equations, derive the Poynting theorem.

(c) Give the physical interpretation of each term in the mathematical expression for the Poynting theorem. What is the physical meaning of the Poynting theorem?

Problem 4

Consider the wave equation $\left(\nabla^2 - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2} \right)\psi = -4\pi f(\vec{\mathbf{x}}, t)$.

(a) Find the corresponding Green function G for a system with no boundary.

(b) Express the formal solution for ψ in terms of G.

(c) For the retarded Green function

$$G(\vec{\mathbf{x}}, t'; \vec{\mathbf{x}}', t') = \frac{1}{R}\delta(t' - t + R/c),$$

where $R = |\vec{\mathbf{x}} - \vec{\mathbf{x}}'|$, provide a physical interpretation for the behavior of the solution of ψ.

(d) Maxwell's equations lead to the following wave equation for the electric field:

$$\nabla^2\vec{\mathbf{E}} - \frac{1}{c^2}\frac{\partial^2\vec{\mathbf{E}}}{\partial t^2} = -\frac{1}{\epsilon_0}\left(-\vec{\nabla}\rho - \frac{1}{c^2}\frac{\partial\vec{\mathbf{J}}}{\partial t} \right)$$

Write down the solution for $\vec{\mathbf{E}}$ in terms of the retarded Green function.

4

A conducting sphere of radius b is maintained at a potential $V(\theta) = V_0 \cos\theta$. It is placed at the center of a grounded hollow sphere of radius a, where $a > b$.

(a) Find the electrostatic potential Φ in the region $b < r < a$.

(b) Find the surface charge densities on the conducting surfaces at $r = a$ and $r = b$.

A thin, spinning sphere of radius a with an uniform surface charge density σ is rotating at a constant angular velocity ω as shown.

The magnetic field $\vec{\mathbf{B}}$ generated in the regions inside and outside the sphere is given by (Gaussian units; $r \equiv |\vec{\mathbf{x}}|$)

$$\vec{\mathbf{B}}(\vec{\mathbf{x}}) = \begin{cases} \dfrac{8\pi a \sigma \omega}{3c}\, \hat{\mathbf{z}}, & r < a, \\[2ex] \dfrac{3\vec{\mathbf{x}}\,(\vec{\mathbf{m}} \cdot \vec{\mathbf{x}}) - r^2 \vec{\mathbf{m}}}{r^5}, & r > a, \end{cases}$$

with

$$\vec{\mathbf{m}} = \frac{4\pi \sigma \omega a^4}{3c}\, \hat{\mathbf{z}}.$$

Find the magnetic force $\vec{\mathbf{F}}$ on the top half of the sphere ($0 < \theta < \frac{\pi}{2}$ for polar angle θ) and show that it is in the $-\hat{\mathbf{z}}$ direction. The force on the bottom half of the sphere opposes this, which indicates that the sphere is "squashed" by the force.

Part IV: Statistical Mechanics & Mathematical Methods
Saturday, August 15, 2015, 1:30 pm

There are six problems, each worth 20 points. After all six problems are graded, the top five scores will be totaled. Maximum points: 100

Problem 1

Consider a two-level system with energy states ϵ and $\epsilon + \Delta$, where $\Delta \geq 0$.

(a)   Compute the partition function and the free energy.

(b)   Derive an expression for the specific heat $C(T)$.

(c)   What are the low-T and high-T limits of this expression? Sketch your result.

Problem 2

(a) Given entropy $S = S(V, T)$ and volume $V = V(P, T)$, the specific heat at constant pressure and volume are defined as follows:

$$c_p \equiv T \left(\frac{\partial S}{\partial T} \right)_P \quad \text{and} \quad c_v \equiv T \left(\frac{\partial S}{\partial T} \right)_V.$$

Show that

$$c_p - c_v = T \left(\frac{\partial S}{\partial V} \right)_T \left(\frac{\partial V}{\partial T} \right)_P.$$

(b) For a monatomic ideal gas, the Sackur-Tetrode equation gives an expression for the entropy:

$$S(U, V, N) = k_B N \left(\ln \left[\frac{V}{N} \left(\frac{4\pi m U}{3 h^2 N} \right)^{3/2} \right] + \frac{5}{2} \right).$$

Use the Sackur-Tetrode equation and the definition of pressure in statistical physics

$$P \equiv T \left(\frac{\partial S}{\partial V} \right)_{U,N}$$

to derive the ideal gas law.

(c) Starting with formulas from parts (a) and (b), determine a simple expression for $c_p - c_v$ for a monatomic ideal gas. Explain the physical significance of $c_p > c_v$.

In a certain system, the internal energy E is related to the entropy S, particle number N, and volume V through

$$E = CN \left(\frac{N}{V}\right)^d e^{\frac{Sd}{Nk_B}}$$

where C is a constant and d is a constant related to the dimensionality.

(a) Calculate the temperature T of the system.

(b) Calculate the Helmholtz free energy A and express it in terms of its native variables V and T, that is, $A(V, T)$.

(c) Calculate the chemical potential of the system.

Problem 4

Using Fourier transform methods, solve the following equation for $f(y)$:

$$\int_{-\infty}^{\infty} dy\, e^{-|x-y|} f(y) = \frac{1}{1+x^2}$$

Problem 5

For the curve $y = \sqrt{x}$ between $x = 0$ and $x = 2$, find the following:

(a) the area under the curve

(b) the length of the curve

(c) the volume of the solid generated when the area under the curve is revolved about the x axis

(d) the outer surface area of the solid shown in the figure, which includes both the curved surface and the circle in the y-z plane at $x = 2$.

Problem 6

(a) Determine the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$.

(b) Using the calculus of residues, evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos mx}{(x^2 + a^2) + b^2}\, dx$$

where a is real, b is real and positive, and m is a positive integer.

Baylor University Physics Ph.D. Program
Preliminary Exam 2016

Part I: Classical Mechanics
Thursday, August 11, 2016, 9:00 am

There are six problems, each worth 20 points. After all six problems are graded, the top five scores will be totaled. Maximum points: 100

Problem 1

A uniform right circular cone of height h and base R has a mass M. It is set on its side and it rolls without slipping in such a way that the tip of the cone remains fixed. Note that, instantaneously, the axis of rotation is along the line of contact between the cone and the horizontal surface. The angle between this line and a fixed line on the horizontal plane is ϕ so the angular velocity of the center of mass of the cone is $\dot{\phi}$.

(a) Determine the inertia tensor relative to a set of principal axes.

(b) Obtain an expression for the kinetic energy of the cone.

Small oscillations of two masses suspended/connected by springs. A massless spring of constant k is fixed to the ceiling and is hanging vertically. A particle of mass m is attached to the bottom of the spring. A second identical spring is suspended from the particle and finally a second particle (also having mass m) is attached to the second spring. The system only moves in the vertical direction. Determine the normal frequencies and the normalized eigenvectors $\mathbf{a_1}$ and $\mathbf{a_2}$.

A speed bump in the road can be modeled as a section of a cylinder, as shown in the figure, where the radius of the cylinder is $R = 2$ meters and the bump subtends a total arc of $44.6°$. Assume that the car has speed v_0 at the top of the bump when the driver takes his or her foot off of the gas pedal at the top of the bump so that the car is coasting. Using the Lagrangian method with undetermined multipliers, estimate the maximum speed v_0 with which a car can be driven over the top of the speed bump without leaving the surface of the bump. As a first approximation, treat the car as a point particle of mass m with no frictional losses.

speed bump

road

$\theta = 22.3°$

R

Problem 4

The figure below shows the collision between masses m_1 and m_2 in the laboratory reference frame. Consider this collision in the center of mass (CM) reference frame with velocities u'_1, u'_2 (before) and v'_1, v'_2 (after).

(a) Show that in the CM frame, both initial total momentum and final total momentum are zero.

(b) For an elastic collision, prove that the individual kinetic energy of each mass in the CM frame is separately conserved. What would your results be if the collision were inelastic?

(c) Use the results in part (a) and (b) to solve the following problem. A horizontally moving cannon (mass M) in the sky with speed v_0 at an elevation H explodes with additional energy E_0 into two fragments (masses $m_1 + m_2 = M$). If both fragments travel in the same horizontal direction after the explosion, find the distance separating the two fragments when they land on the ground.

Problem 5

A block of mass m_1, lying on a frictionless inclined plane of a wedge (mass M), is connected to a mass m_2 by a massless string passing over a pulley (again massless) as shown in the figure below. The wedge is resting on a frictionless horizontal surface. You are asked to use the Lagrangian undetermined multiplier method to determine the acceleration of M, acceleration of m_1 and m_2 system, and the tension of string.

(a) Propose generalized coordinates to find the equations of motion for the problem.

(b) Identify all constraints in the problem and write a constraint equation for each.

(c) Determine the necessary Lagrangian for the problem.

(d) Apply the Lagrangian undetermined multiplier method and obtained the necessary equations of motions.

(e) Solve them to determine the acceleration of M, and the acceleration of the m_1 and m_2 system, and the tension in the string.

(f) If $m_2 = m_1 \sin \theta$, what would be the accelerations and tension of the string? Your answer must be consistent with the result of (e).

Problem 6

A two-dimensional coordinate system that is useful for orbit problems is the *tangential-polar* coordinate system. In this coordinate system, a curve is defined by r, the distance from the origin O to a general point P of the curve, and p, the perpendicular distance from O to the tangent to the curve at P.

(a) Let ϕ denote the angle between the radius vector and the tangent to the orbit at any instant, where the radius vector extends from the origin to the point P. Draw a careful diagram that shows a curve of your choice. Label a point P on the curve, and then label r, p, and ϕ.

(b) Consider a particle of mass m moving under the influence of a force F directed toward the origin O. Using tangential-polar coordinates, prove that

$$F = -mv\frac{dv}{dr} \quad \text{and} \quad mv^2 = Fp\frac{dr}{dp}.$$

(c) Show that $h = mpv$ is a constant of the motion and that

$$F = \frac{h^2}{mp^3}\frac{dp}{dr}.$$

Baylor University Physics Ph.D. Program
Preliminary Exam 2016

Part II: Quantum Mechanics
Thursday, August 11, 2016, 1:30 pm

There are six problems, each worth 20 points. After all six problems are graded, the top five scores will be totaled. Maximum points: 100

Problem 1

A particle is trapped between two impenetrable potential walls at $x = \pm a$.

(a) Find the eigenvalues and corresponding renormalized eigenfunctions.

(b) Calculate the corresponding density and flux.

(c) Calculate the expectation value of the momentum p for each eigenstate, and explain your result in terms of the results obtained in (b).

Problem 2

Consider a particle of mass m that is free to move on a one-dimensional ring of circumference L (as in a bead sliding on a frictionless circular wire).

(a) Find the (normalized) stationary states of the system and their corresponding energies [Note: You should obtain a set of states that can be labeled with a discrete integer index n, where $n = 0, \pm 1, \pm 2, ...$].

(b) Now suppose a perturbation is introduced to the system of the form:

$$H' = -V_0 e^{-x^2/a^2},$$

where $a \ll L$. Using degenerate perturbation theory, find the first-order correction to the energies for the $n \neq 0$ states [Hint: Since $a \ll L$, H is essentially zero outside the range $-a < x < a$; so, integral limits can be extended from $\pm L/2$ to $\pm \infty$.].

(a) Using the trial function,

$$\psi(x) = \frac{A}{x^2 + b^2},$$

find the best bound on the ground state energy of the one-dimensional harmonic oscillator. Simplify your answer as much as possible, and compare your result with the exact value.

(b) Prove the following corollary to the variational principle: If $\psi(x)$ is normalized and $\left\langle \psi \,\middle|\, \psi_{\text{ground state}} \right\rangle = 0$, then $\langle H \rangle \geq E_2$, where E_2 is the the energy of the first excited state.

A spherical square well has a depth V_0 and a radius a. A particle of positive energy E and mass m is caught in the well in a state of angular momentum $L \neq 0$. Using the WKB approximation, the transmission factor T, the ratio of the flux outside the well to inside the well is given by

$$T = \left(\frac{E}{E + V_0} \right)^{1/2} \exp\left[-2 \int k(r)dr \right],$$

where $\frac{\hbar^2 k^2}{2m} = V - E$, and r is the radial variable.

(a) Find the rate at which particle inside the spherical square hits the wall.

(b) Compute the probability that the particle escapes from the spherical potential on a given strike against the well wall.

(c) Estimate the lifetime τ of the particle remaining inside the well. In your estimate, assume the probability of escape is low for a given time the particle strikes the well wall. A related approximation for an integral that may come in handy is

$$\int_\gamma^1 \frac{dx}{x}(1 - x^2)^{1/2} \sim \ln(1/\gamma),$$

for the case of $0 < \gamma << 1$.

Problem 5

In a hydrogen atom, the electron is moving in the excited $l = 3$ orbit.

(a) Write down all of the possible eigenstates of the electron in terms of the operators L^2, S^2, L_z and S_z, where $\mathbf{L}$ and $\mathbf{S}$ denote, respectively, the orbital and spin angular momentum of the electron, and L_z is the z-component of $\mathbf{L}$, and so on.

(b) What are the possible eigenvalues of $\mathbf{J}$ and J_z, where $\mathbf{J} \equiv \mathbf{L} + \mathbf{S}$.

(c) Write down all of the eigenstates of the electron in terms of the operators J^2, L^2, S^2 and J_z.

Two identical particles of spin $1/2$ obey Fermi statistics. They are confined to a cubical box whose sides are 10^{-8} cm in length. There exists an attractive potential between pairs of particles of strength 10^{-3} eV, acting whenever the distance between the two particles is less then 10^{-10} cm. Using non-relativistic perturbation theory, calculate the ground-state energy and wave function. [Take the mass of the individual particles to be the mass of the electron.]

Baylor University Physics Ph.D. Program
Preliminary Exam 2016

Part III: Electricity & Magnetism
Friday, August 12, 2016, 9:00 am

There are six problems, each worth 20 points. After all six problems are graded, the top five scores will be totaled. Maximum points: 100

Problem 1

An electric charge Q is uniformly distributed along a rod of length L. You can ignore the width of the rod.

-L/2   L/2

(a) Find the electric field at distance r from the rod on the bisecting axis of the rod, and show that it approaches the form for a point-like charge Q for large r.

(b) The same rod of length L is now bent into a half-circle shape. (See the figure below.) Find the electric field in the center of the half-circle.

L/π

Problem 2

Using Biot-Savart law, calculate the magnetic field at the center of a uniformly charged spherical shell of radius R and total charge Q, spinning at constant angular velocity ω.

A thin, circular conducting ring of radius a lies fixed in the x-y plane centered on the z axis. It is driven by a power supply such that it carries a constant current I. Another thin conducting ring of radius b, with $b \ll a$, and resistance R is centered on and is normal to the z axis. This second ring is moved along the z axis at constant velocity v such that it's center is located at $z = vt$. Estimate, using whatever approximations you consider appropriate, the following quantities including the full time dependence.

(a) The current in the moving ring.

(b) The force required to keep the ring moving at constant velocity.

Problem 4

An infinite straight wire runs directly above the x-axis at a perpendicular distance z_0 from the x-y plane. The wire contains a uniform charge per unit length of λ. The x-y plane itself is an infinite grounded conducting sheet.

(a) Find the potential in the region above the x-y plane ($z > 0$).

(b) Find the electric field immediately above the x-y plane ($z > 0$).

(c) Find the charge density $\sigma(x, y)$ on the conducting sheet.

(d) Check your answer in part (c) by calculating the charge in an infinite strip of width L in the x-direction and extending to $\pm\infty$ in the y-direction.

The figure below shows a monochromatic electromagnetic plane wave in oblique incidence to the interface of mediums (1) and (2) with angle of incidence θ_I and propagation vector $\mathbf{k}_I$. The resulting reflected and transmitted waves are at angle of reflection θ_R, with the propagation vector $\mathbf{k}_R$, and angle of transmission θ_T, with the propagation vector $\mathbf{k}_T$. The speed and index of refraction of the two mediums are, respectively, v_1, n_1 and v_2, n_2. This incidence wave is polarized in the plane of incidence, the x-z plane, and its electric wave is in the form of $\vec{\tilde{E}}_I(r,t) = \vec{\tilde{E}}_{oI} e^{i(\vec{k}\cdot\vec{r}-\omega t)}$.

(a) Write the form of the incident magnetic wave, reflected electric wave, reflected magnetic wave, transmitted electric wave, and transmitted magnetic wave using the given parameters and amplitudes of reflected $\left(\vec{\tilde{E}}_{oR}\right)$ and transmitted $\left(\vec{\tilde{E}}_{oT}\right)$ electric waves.

(b) Apply the electrodynamic boundary conditions at $z = 0$ and obtain $\tilde{E}_{oR}$ and $\tilde{E}_{oT}$ in terms of $\tilde{E}_{oI}$, other given parameters, and constants $\epsilon_1, \epsilon_2, \mu_1,$ and μ_2. Show that your results agree with

$$\tilde{E}_{oR} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)\tilde{E}_{oI}, \quad \tilde{E}_{oT} = \left(\frac{2}{\alpha + \beta}\right)\tilde{E}_{oI}, \quad \alpha \equiv \frac{\cos\theta_T}{\cos\theta_I}, \quad \beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2},$$

the well known Fresnel's equations.

(c) Use these relationships to explain the condition for Brewster's angle and phase reversal upon reflection.

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Problem 6

A particle with charge e and mass m is undergoing circular motion with radius R in a uniform magnetic field $B = |\vec{B}|$. The radiation energy loss per cycle, E_{cycle}, is given by

$$E_{cycle} = \frac{4\pi e^2}{3R} \left(\frac{E}{mc^2}\right)^4 \beta^3 \quad \text{(Gaussian)},$$

$$E_{cycle} = \frac{e^2}{3\epsilon_0 R} \left(\frac{E}{mc^2}\right)^4 \beta^3 \quad \text{(SI)},$$

where $\beta \equiv |\vec{v}|/c$. Show that the energy loss per cycle may also be expressed as a function of R and B as

$$E_{cycle} = \frac{4\pi e^2 R^2}{3} \left(\frac{eB}{mc^2}\right)^3 \sqrt{1 + \left(\frac{ReB}{mc^2}\right)^2} \quad \text{(Gaussian)},$$

$$E_{cycle} = \frac{e^2 R^2}{3\epsilon_0} \left(\frac{eB}{mc^2}\right)^3 \sqrt{1 + \left(\frac{ReB}{mc^2}\right)^2} \quad \text{(SI)}.$$

Part IV: Statistical Mechanics & Mathematical Methods
Friday, August 12, 2016, 1:30 pm

There are six problems, each worth 20 points. After all six problems are graded, the top five scores will be totaled. Maximum points: 100

Problem 1

A zipper has $N - 1$ links. Each link is either open with energy ε or closed with energy 0. We require, however, that the zipper can only open from the left end, and that link I can only unzip if all the links to the left $(1, 2, ..., I - 1)$ are already open. Each open link has G degenerate states available to it (it can flop around).

(a) Compute the partition function of the zipper at temperature T.

(b) Find the average number of open links any temperature and at low temperature, i.e., in the limit $\varepsilon \gg kT$.

(c) Find the average number of open links at high temperature, i.e., in the limit $\varepsilon \ll kT$.

(d) Is there a special temperature at which something interesting happens at large N? What happens there?

Problem 2

The entropy of a thermodynamic cavity of radiation is given as:

$$S = \frac{4}{3}\sigma V^{1/4}U^{3/4}.$$

(a) Show that the energy density in this cavity obeys a T^4 relationship.

(b) Demonstrate that the radiation pressure is $1/3$ of the energy density.

(c) In a star (which is a particular type of thermodynamic cavity of radiation) what is the role of this radiation pressure?

You are responsible for testing and repairing a set of N photosensor devices. Each device measures an amount of light for $n = 512$ different inputs (channels). If there are any bad channels on the device, it is necessary for you to disassemble the device, repair the bad channels, and reassemble the device. It takes a time α to disassemble the device, β to repair each bad channel, and γ to reassemble the device. (Assume the time required to test the channels is negligible.)

For parts (a) and (b), assume that each of the n channels is bad with a probability p.

(a) What fraction of the N devices do you expect to have to disassemble?

(b) Show that the average time to repair one device is given by

$$\langle t_b \rangle = (\alpha + \gamma) \left[1 - (1 - p)^n \right] + \beta n p.$$

For parts (c) and (d), assume that the number of bad channels in each device follows a Poisson distribution with mean μ.

(c) What fraction of the N devices do you expect to have to disassemble?

(d) Find the average time $\langle t_p \rangle$ to repair one device in terms of α, β, γ, and μ.

(e) Prove that your result for $\langle t_p \rangle$ in part (d) is a good approximation for $\langle t_b \rangle$ when n is sufficiently large and p is sufficiently small. (*Hint:* What is the relationship between μ, n, and p?)

Evaluate the integral

$$I = \int \frac{dz}{4 - z^2}.$$

(a) Along a path from point 0 to i, is the integral dependent of path? If so, state with some examples.

(b) Along a path from point 0 to 1, is the integral dependent of path? If so, state with some examples.

(c) Along the circle C which is $|z - 0.5| = 1$.

(d) Along the circle C which is $|z - i| = 6$.

(e) Along the circle C which is $|z - 3i| = 1$.

Problem 5

Find two power series solutions of the equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 2y = 0$$

by completing the following steps:

(a) Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$, and find the recurrence relation that relates a_m to a_{m-2}.

(b) By setting $a_0 = 1$ and $a_1 = 0$, show that one solution to the equation is

$$y_1(x) = e^{x^2}.$$

(c) By setting $a_0 = 0$ and $a_1 = 1$, find a second solution to the equation, $y_2(x)$. Express your result as an infinite series.

(d) As an added bonus, use $y_2(x) = u(x)y_1(x)$ in the method above to obtain the following useful result:

$$\int_0^x e^{-v^2} dv = e^{-x^2} \sum_{n=0}^{\infty} c_n (2x)^{(2n+1)}$$

Determine c_n.

(Hint: $c_1 = \frac{1}{12}$.)

Problem 6

Calculate the Fourier transform $\hat{f}(\mathbf{k})$ of the three-dimensional function

$$f(r) = \frac{1}{r^2 + \lambda^2},$$

where $\mathbf{r} = (x, y, z)$ and $r^2 = x^2 + y^2 + z^2$.

Baylor University Physics Ph.D. Program Preliminary Exam 2017

Part I: Classical Mechanics

(Monday, August 14, 2017, 9:00am)

There are six problems, each worth 20 points. After all six problems are graded, the top five scores will be totaled. Maximum points: 100

$\boxed{\text{Problem 1}}$

A particle of mass m moves under the influence of the central force

$$\vec{F} = -c^2 \frac{\vec{r}}{r^{5/2}}.$$

a) Calculate the potential energy.

b) Use the Lagrangian approach to show that angular momentum is conserved and identify the (fictitious) effective potential.

c) Find the radius of any circular orbit in terms of the angular momentum.

d) Plot the effective potential and identify the types of orbits.

Problem 2

A system consists of two identical masses m which are connected by three identical springs with spring constant k and equilibrium length b, as shown in the figure.

a) Determine the normal modes and normal coordinates of the system.

b) Suppose that the system is initially at rest, and that a force $F = F_0 \cos \omega t$ is applied to the mass on the right. Find the motion of the system.

Problem 3

A neutron star is so massive and dense that the gravitational force causes all of the protons and electrons to combine into neutrons. The star is supported from further gravitational collapse by degenerate neutron pressure. However, it is not as dense as a black hole, and radiation can escape from its surface.

a) (Ignore relativistic gravitational effects) Develop a model for the frequency f' that we would observe for a photon of frequency f which leaves the surface of a neutron star (mass M and radius R), if the photon were a particle in the Newtonian sense. (You must develop an argument for defining the effective mass of the photon.)

b) Derive a formula to determine the percent difference in the observed frequency, f', calculated using classical physics, as derived in part (a) and the observed frequency calculated using general relativity, $f'' = f\sqrt{1 - \frac{2GM}{rc^2}}$.

c) Estimate this percent difference for a neutron star with a mass equal to that of our sun with a radius of 15 km. ($M_{sun} = 1.99 \times 10^{30}$kg)

Problem 4

A bead with mass m, charge q is constrained to move on a non-conducting, frictionless helical wire such that its path has a fixed radius R and its position along the wire is $z = a\phi$, ϕ being the azimuthal angle. Two fixed masses, each with charge Q, are located at $z = \pm h$.

a) Determine the equation of motion for the bead. (You can neglect the effect of gravity.)

b) Show that there is a stable equilibrium point at $z = 0$.

c) Determine the frequency of small oscillations about this equilibrium point.

Problem 5

1. A small disk of mass m sliding on a frictionless surface collides with a uniform stick, mass M and length L, lying on the surface. The disk is initially traveling with speed v_0 perpendicular to the stick. The disk strikes the stick at a distance d from the center of mass (COM) of the stick. The collision is elastic, and the disk moves in the same direction after the collision.

a) Find the resulting translational and rotational speeds of the stick, and also the resulting speed of the disk. Express your answers in terms of only the given quantities.

b) What can you say about "before and after relative velocities" of 1-D elastic collisions?

c) Prove your statement about relative velocities in part (b) for the collision between the disk and stick in part (a).

Problem 6

1. Consider a thin homogeneous plate that lies in the xy-plane.

a) Show that the inertia tensor takes the form

$$\mathbb{I} = \begin{bmatrix} A & -C & 0 \\ -C & B & 0 \\ 0 & 0 & A+B \end{bmatrix}$$

b) If the coordinate axes are rotated through an angle θ about the z-axis, show that the new inertia tensor is

$$\mathbb{I}' = \begin{bmatrix} A' & -C' & 0 \\ -C' & B' & 0 \\ 0 & 0 & A'+B' \end{bmatrix}$$

And give expressions for A', B', C' in terms of A, B, C and θ.

c) Show that the x and y axes become principal axes if the angle of rotation is

$$\theta = \frac{1}{2} \tan^{-1} \left(\frac{2C}{B-A} \right).$$

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Baylor University Physics Ph.D. Program
Preliminary Exam 2017

Part II: Quantum Mechanics
(Monday, August 14, 2017, 1:30pm)

There are six problems, each worth 20 points. After all six problems are graded, the top five scores will be totaled. Maximum points: 100

Problem 1

Consider an electron in a uniform magnetic field in the positive z-direction. The result of a measurement has shown that the electron spin is along the positive x-direction at $t = 0$. The Hamiltonian is $H = \mu_o B \sigma_z$, where $\mu_o =$ represents the magnetic moment of the electron and B is the magnetic field strength. Use the Pauli spinor basis

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For $t > 0$ compute the respective quantum mechanical probabilities for finding the electron in the state

a) $S_x = \frac{1}{2}$;

b) $S_x = -\frac{1}{2}$;

c) $S_z = \frac{1}{2}$.

Problem 2

A two-dimensional oscillator has the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(1 + \delta xy)(x^2 + y^2).$$

(using $\hbar = 1$). Take $0 \le \delta \ll 1$.

a) Give the wave functions and energy eigenvalues for the first three lowest energy states for $\delta = 0$.

b) Evaluate the first-order perturbations to these energy levels for $0 < \delta \ll 1$.

Problem 3

Consider the potential

$$V(x) = -\frac{\hbar^2 a^2}{m}\operatorname{sech}^2(ax),$$

where a is a positive constant and "sech" stands for the hyperbolic secant.

a) Sketch this potential.

b) Find the energy of the ground state wave function $\psi_0(x) = A\operatorname{sech}(ax)$. Calculate the normalization constant A.

c) Use the WKB method to obtain an approximation to the bound state energy for this potential. Compare with the exact answer.

Hint: The substitution $z = \operatorname{sech}(ax)$ may be useful.

Note: This is a famous example of a reflectionless potential. Every incident particle (with positive energy) passes through without reflection!

Problem 4

A particle of mass m resides in an infinite square well with

$$V(x) = \begin{cases} 0 & 0 < x < a, \\ \infty & \text{otherwise.} \end{cases}$$

It starts out in the left half of the well, and is (at $t = 0$) equally likely to be found at any point in that region.

a) What is the probability that a measurement of the energy would yield the value $\pi^2\hbar^2/(2ma^2)$?

b) A small perturbation to the floor of the well is introduced having the form

$$H' = \frac{V_0}{a^2}\left(ax - x^2\right).$$

Sketch the potential and find the first-order corrections to the allowed energies.

Problem 5

A particle in a spherically symmetrical potential is known to be in an eigenstate $|l, m\rangle$ of L^2 and L_z with eigenvalues $\hbar^2 l(l+1)$ and $m\hbar$, respectively.

a) Prove that $L^2 = L_z^2 + L_+ L_- - \hbar L_z$, where $L_\pm = L_x \pm iL_y$.

b) Derive the coefficient $c_-^{(l,m)}$ that appear in $L_- |l, m\rangle = c_-^{(l,m)} |l, m-1\rangle$.

c) Calculate the expectation values $\langle L_\pm \rangle$, $\langle L_\pm L_\mp \rangle$, $\langle L_x^2 \rangle$ and $\langle L_y^2 \rangle$ of the state $|l, m\rangle$.

Problem 6

Consider the scattering of a particle of mass m, of incoming momentum $\vec{p} = p\hat{z}$ along the z-axis, from a potential $V(r)$ given by

$$V(r) = -\frac{\lambda}{2m}\delta(r-a)$$

where λ is a strength parameter with the dimension $(\text{length})^{-1}$ and here we set $\hbar = 1$.

a) Show that the ℓ-th partial wave scattering amplitude $f_\ell(p)$ is given by

$$\frac{f_\ell(p)}{a} = \frac{e^{\delta_\ell(\xi)}\sin\delta_\ell(\xi)}{\xi} = \frac{g[j_\ell(\xi)]^2}{1 - i\xi g j_\ell(\xi)h_\ell^{(1)}(\xi)},$$

where $\xi = pa$, $g = \lambda a$, and $j_\ell(x)$ and $h_\ell^{(1)}(x)$ are the respective spherical Bessel and Hankel functions. As usual, $\delta_\ell(\xi)$ is the corresponding phase shift.

b) Show that the minimum strength required to bind a state of angular momentum ℓ is $g = 2\ell + 1$.

Baylor University Physics Ph.D. Program
Preliminary Exam 2017

Part III: Electricity & Magnetism
Tuesday, August 15, 2017, 9:00am

There are six problems, each worth 20 points. After all six problems are graded, the top five scores will be totaled. Maximum points: 100

Problem 1

A conducting spherical shell of inner radius a is held at zero potential. The interior of the shell is filled with electric charge of a volume density

$$\rho(\vec{r}) = \rho_0 \left(\frac{a}{r}\right)^2 \sin^2 \theta$$

(a) Find the potential everywhere inside the shell. The Green's function for the inside of the spherical shell is given as,

$$G(\vec{x}, \vec{x}') = 4\pi \sum_{l,m} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} \left(1 - \left(\frac{r_>}{a}\right)^{2l+1}\right) Y_{lm}^*(\Omega') Y_{lm}(\Omega)$$

(b) What is the surface charge density on the inside surface of the shell?

Problem 2

A conducting rod of mass m slides on frictionless, conducting rails whose separation is l in a region of constant magnetic field $\mathbf{B}$ (direction into the page). The rails are connected to a resistor R. Assume that the conducting rod and rails have negligible resistances.

(a) At time $t = 0$, the rod just enters into the region of magnetic field with a constant velocity v_0. Find the velocity of the rod at $t > 0$.

(b) Now a battery with voltage V_0 is connected to the rail in series with the resistor as you find in the following figure. Assume that the rod is at rest at $t = 0$. When the rod is at rest, there is no induced EMF, and the current is purely driven by the battery. This current in turn pushes the rod due to the Lorentz force. Once the rod slides, there is an induced EMF. Find the velocity of the rod $v(t)$ for $t > 0$. In this problem we assume that the applied magnetic field $\mathbf{B}$ is much larger than the magnetic field generated by the rails.

Problem 3

Consider a square loop of width w, carrying a current I.

(a) What is the magnetic dipole moment?

(b) Find the exact magnetic field [hint: don't just consider $\mathbf{B}_{\text{dipole}}$, but get the "exact" magnetic field] a distance z above the center of the loop, and verify that it reduces to the field of a dipole, with the appropriate dipole moment, when $z \gg w$, i.e.

$$\mathbf{B} \approx \frac{\mu_0 \mathbf{m}}{2\pi z^3} \hat{z}$$

A few useful formulas:

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \hat{\mathbf{t}}}{\mathbf{t}^2} dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\mathbf{t}}}{\mathbf{t}^2}, \qquad \mathbf{m} = I \int d\mathbf{a}$$

$\boxed{\text{Problem 4}}$

Starting with Maxwell's equations, obtain an expression describing the propagation of a plane wave of frequency ω in an extended medium of conductivity σ, permittivity ϵ, and permeability μ.

Problem 5

A long copper cylindrical shell is subjected to an increasing magnetic field given by $B = B_z(\text{tesla}) = 2e^{at}$, where $a = 10/\text{sec}$. The cylinder is 0.1 m in radius, 1.26 m high and of a thickness of copper such that its circumferential resistance is 0.87×10^{-6} ohms. What current flows in the cylinder at $t = 10$ msec? [It starts from zero at $t = 0$].

Problem 6

An infinitely long conducting cylinder of radius a is concentric with the z-axis so that its cross section in the $x - y$ plane is centered about the origin $\vec{O}$. On the surface of the cylinder, there are infinitesimal gaps at azimuthal angles $\phi = 0$, $\phi = \pi$ and the electrostatic potential $\Phi(a, \phi, z)$ satisfies,

$$\Phi(a, \phi, z) = \begin{cases} V_0, & 0 < \phi < \pi, \\ -V_0, & \pi < \phi < 2\pi, \end{cases}$$

where V_0 is a constant and $\vec{x} = (\rho, \phi, z)$ are the usual cylindrical coordinates. Find $\Phi(a, \phi, z)$ everywhere inside the cylinder in closed form.

Baylor University Physics Ph.D. Program Preliminary Exam 2017

Part IV: Statistical Mechanics & Mathematical Method

(Tuesday, August 15, 2017, 1:30pm)

There are six problems, each worth 20 points. After all six problems are graded, the top five scores will be totaled. Maximum points: 100

Problem 1

For an interacting gas, the van der Waals equation gives corrections the ideal gas equation of state due to interatomic or intermolecular interactions.

$$\left(P + \frac{an^2}{V^2} \right) (V - nb) = nRT,$$

where n is the number of moles, and R is the ideal gas constant. a accounts for a reduction in pressure due to the interatomic or intermolecular attraction, and b represents the reduction of volume available to the particles due to the repulsive core. You may assume a monatomic gas.

a) Calculate the molar heat capacity difference $c_P - c_V$ and show that it is a function of temperature for the van der Waals gas.

b) Show in the appropriate limit of the van der Waals expression that the ideal gas result is obtained.

Problem 2

The energies of the harmonic oscillator are given by $E_n = (n + 1/2)\, \hbar\omega$ in a system of N particles. The infinite number of states may be numbered by considering two sets, the odd numbered states, and the even numbered states.

a) Determine the probability of finding a particle in any even numbered state as a function of T.

b) Then determine the probability of finding a particle in any odd numbered state as a function of T.

c) Determine the limits of the expressions in a) and b) as T goes to zero, and as T goes to infinity.

d) Discuss why the results in c) are physically expected.

Problem 3

In the grand canonical ensemble, the grand partition function is given by $\mathbb{Q} = \sum_{rs} e^{-\alpha N_r - \beta E_s}$, where $\alpha = -\beta\mu$ and N_r and E_s are the variables.

a) Show that the mean square fluctuation in the particle number N is

$$\overline{(\Delta N)^2} \equiv \overline{N^2} - \overline{N}^2 = kT \left(\frac{\partial \overline{N}}{\partial \mu} \right)_{T,V}$$

b) Similarly, show that

$$\left\{ \overline{(NE)} - \overline{N}\,\overline{E} \right\} = - \left(\frac{\partial \overline{N}}{\partial \beta} \right)_{\alpha,V}$$

c) Using the results above, show that

$$\left\{ \overline{(NE)} - \overline{N}\,\overline{E} \right\} = \left(\frac{\partial U}{\partial \overline{N}} \right)_{T,V} \overline{(\Delta N)^2}$$

where $U \equiv \overline{E}$.

Problem 4

Solve the problem $x'' - 3x' + 2x = h(t)$, $x(0) = 2$, $x'(0) = 0$, where

$$h(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases}$$

Problem 5

Use Green's function to solve the following initial ordinary different equation

$$x'' + 9x = h(t), \quad x(0) = 1, \quad x'(0) = 1,$$

where $h(t) = 0$ for $t < \frac{\pi}{2}$, and $h(t) = \sin(t)$ for $t > \frac{\pi}{2}$.

Problem 6

a) Determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$$

b) Using the calculus of residues, evaluate the integral,

$$\int_{-\infty}^{\infty} \frac{\cos(mx)}{(x+a)^2 + b^2} dx$$

where a is real, b is real and positive, and m is an integer.

Baylor University Physics Ph.D. Program
Preliminary Exam 2018

Part I: Classical Mechanics
(Friday, August 10, 2018, 9:00am)

Problem 1

A flexible cable of uniform mass per unit length ρ and fixed length L is suspended between two supports in a gravitational field g. If $y(x)$ represents the curve of the cable,

(a) Write an expression for the total gravitational potential energy of the cable.

(b) Find the differential equation $y(x)$ that minimizes the total gravitational energy.

(c) If possible, find the solution for $y(x)$ for arbitrarily placed supports (don't evaluate the constants of integration). You may find this useful:

$$y = C\sqrt{1 + (y')^2} \quad \rightarrow \quad y = \cosh\left(\frac{x + C_1}{C}\right)$$

Problem 2

Two masses, each with mass m and charge q, are attached
to identical strings of length b and the system is supported
by a horizontal rod to form two simple pendulums as shown
in the figure. When two strings are vertical the separation
between two masses is l. Consider the small oscillations
(small angle approximation) in this problem.

(a) Find the kinetic energy, potential energy and determine the two matrices (tensors)
needed to obtain the secular equation.

(b) Write the secular equation, find the eiganfrequencies, and show they are in the form

$$\omega_1 = \sqrt{\frac{g}{b}}, \qquad \omega_2 = \sqrt{\frac{g}{b} + \frac{2\beta}{m}}$$

where $\beta = 2kq^2/l^3$ with k as a Coulomb constant.

(c) Find the normal modes and explain the physical meaning of each mode.

(d) If one of the charges is $-q$, what differences would you expect in the normal modes?
Justify your answer.

Problem 3

Two balls of masses m_1 and m_2 are placed on top of each other (m_1 on top of m_2, with a small gap between them) and then dropped from height h onto the ground. The mass m_2 makes an elastic collision from the ground, and m_1 and m_2 make an elastic collision. Neglect air resistance. The height h is substantially larger than the size of the two balls, and the size of the two balls can be neglected.

(a) What is the ratio m_1/m_2 for which the top ball of mass m_1 receives the largest possible fraction of the total energy of the system after the collision?

(b) What is the height of the bounce for the top ball in this part (a) case?

(c) The top ball makes a bound of the maximum height, when $m_2 \gg m_1$. What is the maximum possible height of the bounce for the top ball?

Problem 4

A particle is moving in an orbit for the central force field, described by the equation $r = k\theta^2$.

(a) Sketch the orbit using the positive x-axis as $\theta = 0°$. Include the intercepts at $\theta = 0°$, $90°$, $180°$, $270°$, and $360°$ in your sketch.

(b) Starting from the equation

$$\dot{r}^2 = \frac{2}{\mu}(E - U) - \frac{\ell^2}{\mu^2 r^2},$$

show that

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{2\mu r^4}{\ell^2}\left(E - U - \frac{\ell^2}{2\mu r^2}\right).$$

(c) Using the result in (b), find the potential energy $U(r)$ and the force $F(r)$, which the particle is subjected to.

Problem 5

A projectile is fired to the west from a point on the earths surface at latitude λ in the northern hemisphere. The initial angle of inclination of the trajectory with the horizontal is ϕ.

(a) Find the deviation from the straight-line trajectory due to the rotation of the earth.

(b) Which direction is the deviation?

Problem 6

A particle of mass m rests on a smooth plane. The plane is raised to an inclination angle θ at a constant rate α ($\theta = 0$ at $t = 0$), causing the particle to move down the plane. Determine the motion of the particle.

Baylor University Physics Ph.D. Program
Preliminary Exam 2018

Part II: Quantum Mechanics
(Friday, August 10, 2018, 1:30pm)

Problem 1

A particle with mass m and energy E moves in the one-dimensional potential $V(x)$:

$$V(x) = \begin{cases} 0, & (x < 0) \\ V_0, & (x \geq 0) \end{cases} \quad , \text{ where } V_0 \geq 0$$

(a) Solve the time-independent Schrödinger equation for the wave function $\psi(x)$ at all values of x with the boundary condition that the incident flux is from $x = -\infty$ and $E > V_0$.

(b) Calculate the transmission and reflection probabilities from your results in (a).

(c) What are the transmission and reflection probabilities in the limits $V_0 \to 0$ and $V_0 \to E$?

Problem 2

Consider a two-level system with the eigenkets $|1\rangle$ and $|2\rangle$. The Hamiltonian operator of a particle is given by

$$H = H_0 \left(|1\rangle \langle 2| + |2\rangle \langle 1| \right),$$

where H_0 is a constant.

(a) Find the energy eigenvalues and the corresponding normalized eigenkets.

(b) Support at $t = 0$ the particle is at the state $|1\rangle$. What is the probability for observing the particle on the state $|2\rangle$ at time $t > 0$.

Problem 3

The potential energy of an electron in a one-dimenstional well is:

$$U(x) = \begin{cases} \infty, & (x \leq 0) \\ 0, & (0 < x < L) \\ U_0, & (x \geq L) \end{cases}$$

(a) Assuming $E < U_0$, find an expression for the energy of the electorn in the potential well.

(b) The expression derived in part (a) should be transcendental. Use graphical methods to sketch appropriate solutions. (Hint: Re-write your expression from part (a) in terms of a sine function.)

(c) What conditions must the roots satisfy?

(d) What are the conditions for there to be at least one bound state?

Problem 4

To the first order in perturbation theory, calculate the correction $E^{(1)}$, to the ground state energy $E^{(0)} = -\frac{Z^2 me^4}{2\hbar^2} = -Z^2$ 13.7 eV of a hydrogen-like atom with nuclear charge Ze, due to the finite spatial extension of the nucleus. Recall that the first-order correction to the ground state energy is

$$E_0^{(1)} = \int \psi_{100}^* H' \psi_{100}^* dv, \tag{1}$$

where H' is the perturbation potential, and $\psi_{100} = \frac{1}{\sqrt{\pi a^3}} e^{r/a}$ is the ground state wave function, with $a = \frac{\hbar^2}{Zme^2}$. For simplicity assume that the nucleus is spherical of radius R. The nuclear charge is uniformly distributed just on the surface of the nucleus and the potential energy of the electron at a radius r can be written as:

$$V(r) = -\frac{Ze^2}{R}, \quad r < R$$
$$= -\frac{Ze^2}{r}, \quad r \geq R$$

(a) Find $E_S^{(1)}$. [Hint: As a first step, determine H_S' (subscript S denoting perturbation due to charge on surface of nucleus). As a second step, express $E_S^{(1)}$ as an integral, and specifying integration bounds. Then compute the integral (referencing Schaum's).]

(b) For $Z = 100.$, which yields $a \approx 5.00 \times 10^{-11}$ cm. and $a/R \approx 50.$, what is the percent first order perturbation $E_S^{(1)}/E^{(0)}$?

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Problem 5

(a) Consider the finite square-well with

$$V(x) = \begin{cases} 0, & |x| < a/2 \\ V_0, & \text{otherwise} \end{cases}$$

Use the WKB method to determine the approximate bound state energy levels.

(b) Suppose an electron is in the ground state of the system. Now introduce a perturbation $-e\,E_{\text{ext}}\,x$ corresponding to an electric field in the $-x$ direction. Sketch the total potential, noting the limits of the tunneling barrier in the $+x$ direction.

(c) Within the WKB approximation, find the transmission probability for the electron to escape the well when it collides with the tunneling barrier.

Problem 6

Assume that $|lm\rangle$ is the eigenfunction of the angular momentum operators L^2 and L_z, with their engenvalues l and m, respectively.

(a) Calculate the expectation values $\langle L_z \rangle$ and $\langle L_y \rangle$.

(b) Calculate the expectation values $\langle L_x^2 \rangle$ and $\langle L_y^2 \rangle$ and then check the uncertainty relations.

Baylor University Physics Ph.D. Program
Preliminary Exam 2018

Part III: Electricity & Magnetism
(Saturday, August 11, 2018, 9:00am)

Problem 1

Consider two parallel infinite flat perfect conductors (1) and (2) of uniform thicknesses a_1 and a_2, respectively, separated by a distance L between their adjacent sides, each parallel to the x-y plane. Thus, if the conductor (1) has its adjacent side in the x-y plane, then the adjacent side of the conductor (2) lies in the plane at $z = L$. Per unit area, conductor (i) has total electric charge Q_i, $i = 1$, 2.

(a) Show that the surface charges on the adjacent surfaces are equal in magnitude but opposite in sign and that the surface densities on the outer surfaces are equal.

(b) Determine the values of the surface charge densities on the adjacent and outer surfaces of the conductors in terms of the Q_i.

Problem 2

A spherical conductor, of radius a, carries a charge Q as shown below. It is surrounded by linear dielectric material of susceptibility χ_e, out to radius b. Note: $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$, $\epsilon = \epsilon_0(1+\chi_e)$.

(a) Find (1) electric field, (2) electric displacement, and (3) polarization $\mathbf{P}$ in the sphere, dielectric material, and outside.

(b) Find the bound charge σ_b and ρ_b for the dielectric material in this configuration.

(c) Find the energy of this configuration. Note: $W = \frac{1}{2} \int (\mathbf{D} \cdot \mathbf{E}) d\tau$.

Problem 3

$\rho(\vec{r}) = As^n$ A R n

s z

σ $s = R$

A flat phonograph record is smeared with a uniform surface charge density σ in the planar region from $a < s < b$, where s is the distance from the z-axis. It rotates at a constant angular velocity ω in the x-y plane. Taking your origin of coordinates at the center of the disk, find the approximate magnetic field produced at distances $|z| \gg b$ along the z-axis

$+z$

ω

b

a

σ

$a < s < b$ s z

Problem 4

(a) Using the expansion

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r'^l}{r^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi)$$

develop the multipole expansion of the potential $\Phi(\vec{x})$ due to a localized charge distribution $\rho(\vec{x})$ in terms of the multipole moments q_{lm} of ρ. Discuss how and under what conditions this expansion can be used to simplify a problem.

(b) Show that, if the charge distribution has axial symmetry (i.e., the charge distribution is invariant under rotations about the z-axis), then the only non-zero multipole moments are q_{l0}.

Useful equation:

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) e^{im\phi}$$

(c) Using the above results, for two point charges q and $-q$ placed on the z-axis at $z = a$ and $z = -a$, compute the non-vanishing component of the dipole moment (given $Y_{10} = \left(\sqrt{\frac{3}{4\pi}} \cos \theta\right)$).

Problem 5

A sphere with radius R consists of uniform linear magnetic material with permeability μ. It is placed in an otherwise uniform background magnetic field $\vec{B}_0$. There are no free currents in or on the sphere.

(a) Show that one is allowed to introduce a magnetic scalar potential,

$$\vec{H}(\vec{x}) \equiv -\vec{\nabla} W(\vec{x})$$

and demonstrate that $W(\vec{x})$ satisfies the Laplace equation

$$\nabla^2 W(\vec{x}) = 0,$$

both inside and outside the material.

(b) Solving the Laplace equation using azimuthal symmetry, and applying boundary conditions at the spheres surface, find $\vec{B}$ inside the sphere in terms of the background field $\vec{B}_0$. Note that the general solution of the Laplace equation with the azimuthal symmetry can be written as:

$$W(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta).$$

Problem 6

Consider a classical electron of charge e and mass m_e (and no spin) moving with speed v in a circular orbit of radius R around a positive charge q, as shown in the figure. A uniform magnetic field B in direction perpendicular to the plane of the orbit is then turned on.

(a) Find the change in the speed of this electron, Δv, due to the electric field generated when the magnetic field is turned on. Assume R is unchanged. Does this electron speed up or slow down?

(b) Assuming there are n such electrons per unit volume, give an expression for the magnetic susceptibility per unit volume. Assume all orbits are perpendicular to the magnetic field, and ignore the difference between B and H.

(c) Give an argument for why it is reasonable to ignore any change in R when the magnetic field is applied as long as $\Delta v/v \ll 1$.

Baylor University Physics Ph.D. Program
Preliminary Exam 2018

Part IV: Statistical Mechanics & Mathematical Method
(Saturday, August 11, 2018, 1:30pm)

Problem 1

For ideal Bose gas,

$$n = \frac{N}{V} = \left(\frac{2\pi m k_B T}{h}\right)^{\frac{3}{2}} g_{\frac{3}{2}}(z); \qquad \frac{P}{k_B T} = \left(\frac{2\pi m k_B T}{h}\right)^{\frac{3}{2}} g_{\frac{5}{2}}(z)$$

where $g_\alpha(z)$ is the Bose-Einstein function of order α

$$g_\alpha(z) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{x^{\alpha-1} dx}{z^{-1} e^x - 1} = z + \frac{z^2}{2^\alpha} + \frac{z^3}{3^\alpha} + \cdots.$$

(a) Treating T and z as independent variables, calculate dn and dP such that

$$dn = AdT + Bdz; \qquad dP = CdT + Ddz$$

(b) Show that the isothermal compressibility κ_T can be written as the particle density $n = N/V$

$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T = \frac{1}{nk_B T}\frac{g_{\frac{1}{2}}}{g_{\frac{3}{2}}}$$

(c) Comment on the behavior of the compressibility as T approaches the characteristic temperature T_c.

Problem 2

2D surface waves have a dispersion relationship given by

$$\omega(k) = (\alpha k^3)^{\frac{1}{2}}$$

If the energy of each excited wave is given by $E_k = \hbar\omega(k)$, determine the temperature dependence of the thermal contribution to the surface energy per unit area at temperature, T. You may leave your answer in terms of a dimensionless integral but you must show the closed form of the temperature dependence.

Hint: The density of states in 2D may be written as $D(k)dk = (Ak)/(2\pi)\,dk$ where A is the area in question.

Problem 3

A system with two nondegenerate energy levels, E_0 and E_1 ($E_1 > E_0 > 0$) is populated by N distinguishable particles at temperature T.

(a) What is the average energy per particle? Express answer in terms of E_0, E_1 and $\Delta E = E_1 - E_0$.

(b) What is the average energy per particle as $T \to 0$? Express answer in terms of E_1 and ΔE.

(c) What is the average energy per particle as $T \to \infty$? Express answer in terms of $E_0 + E_1$ and ΔE.

(d) What is the specific heat at constant volume, c_V, of this system? Express answer in terms of ΔE.

(e) Compute c_V in the limits $T \to 0$ and $T \to \infty$ and make a sketch of c_V versus $\Delta E / k_B T$.

Problem 4

Consider the following equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0, \qquad u(0,t) = u(L,t) = 0, \qquad u(x,0) = u_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = 0$$

Develop the Green's function $G(x|\xi;t)$ yielding the solution

$$u(x,t) = \int_0^L G(x|\xi;t)u_0(\xi)d\xi.$$

Problem 5

Consider a Frobenius power series solution of the Laguerre equation,

$$xy'' + (1 - x)y' + \lambda y = 0,$$

in the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+k}.$$

(a) Using the power series in x, for $0 \le x < \infty$, determine the indicial polynomial and solve for k.

(b) Find the recursion relation for the coefficients a_n.

(c) Show that the solution of the differential equation can be written as

$$y(x) = a_0 \sum_{n=0}^{\lambda} c_{n,\lambda}\, x^n. \qquad \text{Determine } c_{n,\lambda}.$$

(d) What values of λ will cause the solutions to reduce to finite polynomials, thereby ensuring their convergence for $x < \infty$?

Problem 6

Evaluate the open contour integral

$$I = \int_{C_o} \frac{dz}{z^2 + z},$$ (1)

where the curve C_o in the complex z-plane is given by

$$C_o = \{z : |z| = 1 \text{ and } z \neq -1\}$$

and the integral is taken in the counterclockwise direction on the curve.

Baylor University Physics Ph.D. Program
Preliminary Exam 2019

Part I: Classical Mechanics
(Monday, August 12, 2019, 9:00am)

Problem 1

Two balls of masses m and $3m$ collide head-on. The initial speed of the ball of mass m is v, and the second ball is initially at rest.

(a) What are the final velocities of the balls, if the collision is purely elastic and one-dimensional?

(b) What are the final velocities of the balls, if the collision is completely inelastic?

(c) Consider that now the collision is 2-dimensional but not necessarily elastic. If the scatter angles for the two masses with respect to the initial direction of mass m are 45° for mass m and -45° for mass $3m$ (see figure), what are the magnitudes of the velocities for the two masses? Is the collision elastic? If not, what fraction of the energy is "lost" (i.e. no longer kinetic energy)?

Problem 2

Given the differential equations $(s > 0,\quad m > 0,\quad 0 < \epsilon < m)$,

$$m\,\ddot{q}_1 + \epsilon\,\ddot{q}_2 + s\,q_1 = 0, \quad \epsilon\,\ddot{q}_1 + m\,\ddot{q}_2 + s\,q_2 = 0.$$

(a) Solve for the squared eigenfrequencies, ω^2, of the motion.

(b) Find the time-independent ratio of the generalized coordinates, $q_1(t)/q_2(t)$, for each normal mode.

Problem 3

A block of mass m_1 lying on a frictionless inclined plane of a wedge (mass M) is connected to mass m_2 by a massless string passing over a massless frictionless pulley as shown in the Figure. The complete system is placed on a frictionless horizontal surface where wedge is free to move. You are asked to determine the acceleration of masses and the tension of the string using the Lagrangian undermined multiplier method.

(a) Propose a proper set of generalized coordinates to solve the problem.

(b) Identify all constraints in the problem and write a constraint equation for each.

(c) Determine the kinetic and potential energy for the system and find the Lagrangian.

(d) Find the equations of motion for all three masses.

(e) Solve them to determine the acceleration of m_1 and m_2 system on the wedge, the acceleration of wedge and the tension of the string.

(f) Use your results in part (e) to verify the accelerations and tension when $m_2 = m_1 \sin\theta$.

Problem 4

A proto-comet of mass m in the Oort cloud, very far from the sun (mass $M \gg m$), is perturbed such that it heads toward the sun with some initial velocity $v > 0$ and impact parameter b. Ignore any gravitational interactions other than that between the comet and sun.

(a) Write down the effective potential as a function of the comet-sun distance r in terms of the angular momentum L of the comet, the gravitational constant G, and other given quantities.

(b) Use the effective potential to determine the distance of the comet to the sun at its point of closest approach r_0.

(c) Show that the velocity of the comet at the point of closest approach v_{max} is given by

$$v_{\text{max}} = \frac{v^3 b}{GM} \left[\sqrt{1 + \left(\frac{v^2 b}{GM} \right)^2} - 1 \right]^{-1}.$$

Problem 5

While making cupcakes, you accidentally spill flour onto the very top of an inverted hemi-spherical bowl (radius R) sitting on your kitchen countertop. Afterwards, you notice that the flour has left a halo around the bowl, and there is a region near the bowl with very little flour. [For the purpose of this analysis, assume that the flour has a perfectly inelastic collision with the top of the bowl and then slides down the side of the bowl without friction.]

(a) Conceptually, explain why there is a separation of the ring of flour from the rim of the bowl.

(b) Show that the point of departure of the flour from the bowl is given by

$$\cos\theta = 2/3$$

and that the angular velocity is

$$\dot\theta = \sqrt{\frac{2g}{3R}}.$$

(c) From the results of part (b), determine the distance Δr from the base of the bowl to the halo of flour.

Problem 6

(a) Find the inertia tensor for a book (a uniform rectangular parallelpiped) of mass M and width a, thickness b, and height c rotating about a corner (point O in the figure). Use axes parallel to the books edges $(b < a < c)$.

(b) Using symmetry, make an argument for principal axes for which the inertia tensor is diagonal and find the principal moments of inertia.

(c) If you toss the book up in the air so that it is rotating about one of the three principal axes, the motion is stable when the rotation is about two of the three axes, but the motion is unstable (the book tumbles) when the initial motion is about the third axis. Which of the three axes has unstable motion, and why?

Baylor University Physics Ph.D. Program
Preliminary Exam 2019

Part II: Quantum Mechanics
(Monday, August 12, 2019, 1:30pm)

Problem 1

A particle is in the ground state of a symmetric infinite square well with $V(x) = 0$ for $-a/2 < x < +a/2$, and infinite elsewhere.

(a) The well then undergoes an instantaneous symmetric expansion to $-a < x < +a$. Calculate the probabilities of the particle being found in each of the three lowest energy states of the larger well.

(b) Instead, suppose that the well expansion takes place adiabatically. Again, calculate the probabilities of the particle being found in each of the three lowest energy states of the larger well.

Problem 2

An electron is in the spin state: $\chi = A \begin{pmatrix} 1 - 2i \\ 2 \end{pmatrix}$.

(a) Determine the constant A by normalizing χ.

(b) If you measure S_z on this electron, what values could you get and what is the probability for each?

(c) What is $\langle S_z \rangle$ for this spin state?

(d) What is $\langle S_x \rangle$ for this spin state?

Problem 3

For a particle of mass m, consider a Morse potential of

$$V(x) = -\frac{V_0}{\cosh^2(\beta x)},$$

where $V_0 > 0$ and $\beta > 0$.

(a) Illustrate this potential graphically as a function of x.

(b) Write the WKB quantization condition:

$$\frac{1}{\hbar} \int_{x_{\min}}^{x_{\max}} p(x)dx = \left(n + \frac{1}{2}\right)\pi, \quad n = 0, 1, 2, 3, , \ldots.$$

in terms of the bound state energies E_n and $V(x)$. What are $x_{\min}$ and $x_{\max}$ in this case, and what is the physical meaning/interpretation of $x_{\min}$ and $x_{\max}$?

(c) Use WKB methods to determine the particle's Schrödinger equation bound state energies E_n, $n = 0, 1, 2, \ldots$. You might consider the transformation of $z = 1/\cosh^2(\beta x) = \operatorname{sech}^2(\beta x)$ and $b = -E_n/V_0$ to simpify your calculations.

Problem 4

The ground state of the hydrogen atom has electronic wave function $\psi \sim e^{-r/a_0}$, where a_0 is the Bohr radius.

(a) Find the expectation values $\langle r \rangle$, $\langle r^2 \rangle$, and $\langle x^2 \rangle$.

(b) Explain why $\langle r \rangle \neq a_0$.

(c) Determine the most probable radial distance r for an electron in the ground state.

Problem 5

From the conservation law $\partial\rho/\partial t + \vec{\nabla}\cdot\vec{j} = 0$,

(a) derive the expression of $\vec{j}$, where $\rho \equiv |\psi|^2$

(b) prove that the integration $\int \rho(t,\vec{r})d^3x$ is independent of time.

Recall the Schrödingier equation:

$$i\hbar\frac{\partial}{\partial t}\psi(t,\vec{r}) = -\frac{\hbar^2}{2m}\nabla^2\psi(t,\vec{r}) + V(t,\vec{r})\psi(t,\vec{r}),$$

Problem 6

Estimate the ground-state energy of a one-dimensional simple harmonic oscillator using $\langle x|\tilde{0}\rangle = e^{-\alpha|x|}$ as a trial function with α to be varied. For a simple harmonic oscillator we have $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$. Recall that, for the variational method, the trial function gives an expectation value of H such that $\frac{\langle\tilde{0}|H|\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle} \geq E_0$, where E_0 is the ground state energy. You may use:

$$\int_0^\infty e^{-ax}x^n dx = \frac{n!}{a^{n+1}}, \quad |x| = xH(x) - x[1 - H(x)], \quad \frac{dH(x)}{dx} = \delta(x),$$

where $H(x)$ and $\delta(x)$ denote the Heaviside step function and Dirac delta function respectively. (Don't get confused with the Hamiltonian H and the Heaviside step function $H(x)$.)

Baylor University Physics Ph.D. Program
Preliminary Exam 2019

Part III: Electricity & Magnetism
(Tuesday, August 13, 2019, 9:00am)

Problem 1

Find the following electrid fields at:

(a) a distance z above the center of a straight line segment of length $2L$ that acrries a uniform line charge λ

(b) a distance z above the center of a circular loop of radius r that caries a uniform line charge λ

You can use the integral $\int \frac{1}{(x^2+a^2)^{3/2}} dx = \frac{1}{a^2} \frac{x}{\sqrt{x^2+a^2}}$.

Uniform line charge

Problem 2

(a) A long cylinder of radius R has a volume charge density $\rho(\vec{r}) = As^n$ where A is a constant, n is a positive power and s is the distance to the z-axis. In addition, the cylinder has a uniform surface charge density σ located on its surface at $s = R$. Find the electric field everywhere.

(b) Find the force per unit area on the surface of the cylinder due to the charge configuration.

Problem 3

A point charge q is located in free space a distance d from the center of a dielectric sphere of radius a $(a < d)$ and dielectric constant ϵ/ϵ_0.

(a) Write down the electrostatic Maxwell equations inside and outside of the sphere.

(b) Characterize the potential at all points in space as an expansion in Legendre polynomials.

(c) Using boundary conditions, determine the coefficients of the above expansion.

Problem 4

A magnetically "hard" material is in the shape of a right circular cylinder of length L and radius a. The cylinder has a permanent magnetization $\mathbf{M_0}$, uniform throughout its volume and parallel to its axis.

(a) Determine the magnetic scalar potential **on the axis** of the cylinder:

$$\Phi_{\mathrm{M}}(\boldsymbol{x}) = -\frac{1}{4\pi} \int_V \frac{\boldsymbol{\nabla}' \cdot \mathbf{M}(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3 x' + \frac{1}{4\pi} \oint_S \frac{\mathbf{n}' \cdot \mathbf{M}(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} da'.$$

(b) Determine the magnetic field $\mathbf{H}$ and magnetic induction $\mathbf{B}$ at all points on the axis of the cylinder, both inside and outside.

Problem 5

A circular disk of radius R and mass M carries n point charges (q), attached at regular intervals around its rim. The disk lies in the x-y plane with its center on the z-axis. At time $t = 0$ all the charges and disk have the same initial velocity along the z-axis and are also rotating about the z-axis with angular velocity ω_0 when it is released at the origin. The disk is immersed in a time-independent external magnetic field

$$\boldsymbol{B}(s, z) = k(-s\hat{\boldsymbol{s}} + 2z\hat{\boldsymbol{z}}),$$

where k is constant and s is the variable which represents radial distance in cylindrical coordinates. Ignore gravity in the following discussion.
$$\mathbf{B}(s, z) = k\left(-s\hat{\mathbf{s}} + 2z\hat{\mathbf{z}}\right),$$

(a) Find the force that is exerted on each of the charges and the total force that is exerted on the ring.

(b) Find the total torque that is exerted on the ring.

(c) Find the position of the center of the ring, $z(t)$, and its angular velocity, $\omega(t)$, as functions of time. (Note: $d\omega/dt = 0$ at $t = 0$.)

(d) Describe the motion and check that the total energy is constant, confirming that magnetic forces do no work.

Problem 6

(a) Write down the macroscopic Maxwell equations in terms of free charge ρ and free volume current density $\vec{J}$.

(b) Show that Maxwell equations are consistent with the conservation of electric charge.

(c) Drive the wave equation satisfied by the vector potential $\vec{A}$ in Lorenz gauge from vacuum form of Maxwell equations. (Hint: Lorentz gauge is $\boldsymbol{\nabla} \cdot \mathbf{A} + \dfrac{1}{c^2}\dfrac{\partial \Phi}{\partial t} = 0$.)

Baylor University Physics Ph.D. Program
Preliminary Exam 2019

Part IV: Statistical Mechanics & Mathematical Method
(Tuesday, August 13, 2019, 1:30pm)

$\boxed{\text{Problem 1}}$

Using the probability to find a particle having a velocity between v and $v + dv$ given by

$$P(v)dv = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (m\beta)^{\frac{3}{2}} v^2 e^{-\frac{1}{2}\beta m v^2} dv$$

(a) calculate the most probable velocity, (b) the average velocity, and compare them.

Problem 2

Consider extremely relativistic bosons ($E = pc = \hbar kc$) with spin $S = 1$ in a two-dimensional plane.

(a) Show that the density of states for the system is $g(E) = \frac{3}{2\pi(\hbar c)^2}E$.

(b) Show that the relativistic 2D bosons can undergo the BE condensation and calculate the characteristic temperature T_c.

Problem 3

Argon can sublimate directly from its solid state to its gaseous state without becoming a liquid. If the partition function of the gas is given by:

$$Z_{gas} = \frac{Z_{trans}^{Ng}}{N_g!}$$

where Z_{trans} is the partition function of 3D translational kinetic energy states and the partition function of the solid is given by:

$$Z_{solid} = e^{-\epsilon N_s/k_B T} \left[1 / \left(1 - e^{-\frac{\hbar \omega_E}{k_B T}} \right) \right]^{3N_s}$$

where ω_E is the Einstein frequency in the solid, determine the equilibrium pressure of the solid and gas in coexistence at temperature T assuming the gas is ideal and that the total of the particles N_g in the gas and N_s in the solid is N.

Problem 4

Show that

$$\int_0^{2\pi} \frac{d\theta}{(5 - 3\sin\theta)^2} = \frac{5\pi}{32}.$$

Problem 5

The ends and sides of a thin copper bar of length 2 are insulated so that no heat can pass through them. Let $u(x,t)$ denote the temperature in the rod at the point x at time t. The function satisfies the partial differential equation

$$\frac{\partial u}{\partial t} = 1.14\,\frac{\partial^2 u}{\partial x^2}.$$

Find the temperature $u(x,t)$ in the bar if the initial temperature is

$$u(x,0) = 70\sin x, \quad 0 \le x \le 2.$$

Problem 6

The Bessel function $J_n(x)$ is given by the series expansion

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k+1)} \frac{x^{n+2k}}{2^{n+2k}}$$

where Γ is the gamma function, for which $\Gamma(z+1) = z\Gamma(z)$.

(a) Show that $\quad \dfrac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)$.

(b) Using $\quad \dfrac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x) \quad$ and your result from part (a), prove that

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x),$$

where the prime indicates the derivative with respect to x.

(c) Given that $\Gamma(1/2) = \sqrt{\pi}$, show that

$$J_{1/2}(x) = C \frac{\sin x}{\sqrt{x}},$$

and find the constant C.

Baylor University Physics Ph.D. Program

Preliminary Exam 2020

Part I: Classical Mechanics

1. A bucket of water (total mass = 5kg) on the end of a string (1-m-long) (negligible mass) is swung in a vertical circle.

(a) Next to the figure above, draw free body (force) diagrams for the instant when the bucket is on top of its arc and for the instant when the bucket is at the bottom of its arc.

(b) Determine the **minimum speed** the bucket must have at the top of its arc so that the bucket continues moving in a circle.

(c) Will the water fall out of the bucket if the bucket is moving in a speed of 5 m/s? Explain?

(d) Calculate the tension in the string at the bottom of the arc, assuming the bucket is moving at twice the speed of part (b).

2. Consider a circular disc of radius R and infinitesimal thickness s (see figure), whose density varies with the radial distance as $\rho(r) = kr^2$.

(a) Derive expressions for the total mass M of the disc and its moment of inertia I around its symmetry axis. Express I in terms of M and R.

(b) Suppose the disc rotates at a constant angular velocity. What is the kinetic energy T of the disc in terms of M and the velocity v of the outer edge?

Now suppose the disc is converted into a frictionless pulley. A string of length ℓ is run across it, which connects two blocks of masses m_1 and m_2 where $m_1 > m_2$. Assume the string is massless, does not stretch, and does not slip on the pulley. The blocks can be treated as point-like, only move in the vertical direction, and are placed in a gravity field with a downward acceleration g (see figure).

(c) Write down a Lagrangian for this system in terms of the vertical distance $x(t)$ from the center of the pulley (measured positively downwards) to the block having mass m_1. You can assume that the potential energy vanishes at $x = 0$. Do you expect energy to be conserved in this system? Explain your answer.

(d) Write down the Euler-Lagrange equation for $x = x(t)$ and derive an equation for the acceleration $\ddot{x}$. Express your result in terms of m_1, m_2, M, and g.

(e) What is the tension T_1 in the string just above the block with mass m_1? What is the tension T_2 in the string just above the block with mass m_2? What is the net torque τ applied to the pulley? Express your results in terms of m_1, m_2, M, g, and R.

3. (a) Find the inertia tensor for a thin disk with radius R. Each half of the disk (split along the diameter) is made of a different material with mass densities ρ_1 and ρ_2, as indicated in the figure (x-axis coming out of the page). The angle β is arbitrary.

(b) Find the inertia tensor in a new coordinate system, where the disk is rotating about an axis x' which makes an angle α with respect to the x-axis passing through the center of the disk and normal to the plane of the disk. (Specify the three axes of the new coordinate system.)

4. The main character (of mass ~50 kg) in an action movie escapes certain death when falling from a great height by holding out the edges of a coat to use as a parachute. Never mind the gymnastics needed to remove a coat while falling and stretch it out taut, let's estimate how well this idea would work. Use the following assumptions:

- the cloth of the coat allows 50% of the air to pass through it, while 50% of the air molecules are reflected elastically

- the area of the stretched-out coat is $A = 1$ m^2, and is held perpendicular to the direction of gravity

- the composition of air is 78% nitrogen (^{14}N), 21% oxygen (^{16}O), and 1% argon (^{40}Ar) and air can be treated as an ideal gas

- the height from which the character is falling is small enough that you do not need to consider the change of air density with altitude.

(a) Derive an expression to estimate the drag force as a function of the density of the air ρ, the area of the coat A, and the speed at which the character is falling v.

(b) Obtain an expression for the terminal velocity and give a numerical estimate (in m/s) of this speed.

(c) If the character were falling without drag, find the height which would give the same velocity for impact with the ground as that found for the terminal velocity in (b). Comment on the feasibility of the character landing safely using the parachute coat.

5. A uniform chain, of length l_0, with contiguous links has a portion (of length l) hanging over the edge of a smooth table as shown in the Figure (i).

(a) If the chain starts to move from rest, determine the speed of it when the end A leaves the table.

(b) How long does it take to leave the table?

(c) The moment the end A leaves the edge, you catch it from that end and start pulling upward with a force F. What would be the tension of a point at a distance $l_0/3$ from A?

A

(i)

l

B

Baylor University Physics Ph.D. Program
Preliminary Exam 2020

Part II: Quantum Mechanics

Problem 1

In a spin-1 system, an observable Q has the following matrix representation,

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Find the eigenvalues of Q and the corresponding normalized eigenvectors. Is there any degeneracy?

Problem 2

Given a wave function ψ satisfying:

$$H\psi = E\psi$$

and operators a_+ and a_- with the property that:

$$a_-a_+ = \frac{1}{\hbar\omega}H + \frac{1}{2}, \quad \text{and}$$

$$a_+a_- = \frac{1}{\hbar\omega}H - \frac{1}{2}$$

prove that:

$$H(a_+\psi) = (E + \hbar\omega)a_+\psi, \quad \text{and}$$

$$H(a_-\psi) = (E - \hbar\omega)a_-\psi.$$

Problem 3

(a) Consider two states, described respectively by the kets $|\Phi\rangle$ and $|\Psi\rangle$. Assume that $c_n \equiv \langle a^{(n)}| \Phi\rangle$ and $d_n \equiv \langle a^{(n)}| \Psi\rangle$ are all known, where $|a^{(n)}\rangle$ form a complete set of the base kets, with $n = 1, 2, ..., N$, and N denotes the dimensions of the ket space. Find the matrix representation of the operator $A \equiv |\Phi\rangle \langle\Psi|$ in the base $|a^{(n)}\rangle$.

(b) Now consider a spin $\frac{1}{2}$ system. Calculate the representations of the operators $S_+ = \hbar|+\rangle\langle-|$ and $S_- = \hbar|-\rangle\langle+|$ in the base $|S_z; \pm\rangle \equiv |\pm\rangle$.

Problem 4

Recall that for a three-dimensional spherically symmetric potential $V(r)$ (where r is the magnitude of the radial vector $\vec{r}$), the radial wave function $R(r)$ can be expressed as $R(r) = u(r)/r$. The radial Schrödinger equation for $u(r)$ becomes

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + [V + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}]u = Eu$$

Consider a particle of mass m moving in a spherically symmetric potential of the form $V(r) = -V_0$ (with $V_0 > 0$) when $r \le a$ and $V(r) = 0$ when $r > 0$. Find the smallest value of $V_0 > 0$ such that there is a bound state of **zero energy** and **zero angular momentum**.

Problem 5

Consider the angular momentum states $\psi_\ell^{m_\ell}$ and $\psi_s^{m_s}$ where $\ell = 3$ and $s = \frac{1}{2}$. Construct the state $\psi_{\ell s \frac{5}{2}}^{\frac{5}{2}}$, which has total angular momentum quantum number $J = \frac{5}{2}$ and 3-component quantum number $m_J = \frac{5}{2}$, as a linear combination of product states $\psi_\ell^{m_\ell} \psi_s^{m_s} \equiv \psi_{\ell s}^{m_\ell m_s}$, $\ell = 3$, $s = \frac{1}{2}$.

Baylor University Physics Ph.D. Program
Preliminary Exam 2020

Part III: Electricity & Magnetism

Problem 1

Find the force on a square loop placed near an infinite straight wire. Both the loop and the wire carry a steady current I.

Problem 2

A metal sphere of radius R, carrying charge q, is surrounded by a thick concentric metal shell (inner radius a, outer radius b, as in the figure). The shell carries no net charge.

a) Find the electric field in the four regions of (1) $r < R$, (2) $R < r < a$, (3) $a < r < b$, and (4) $r > b$.

b) Find the surface charge density σ at R, at a, and at b.

c) Find the potential at the center, using infinity as the reference point with potential $V = 0$. You may benefit from the electric fields you obtained in the part (a).

d) Now the outer surface is touched to a grounding wire, which drains off charge and lowers its potential to zero (same as infinity). How do your answers to (b) and (c) change? Explain.

Problem 3

Consider a particle of charge q and mass m, free to move in the xy plane in response to an electromagnetic wave propagating in the z direction:

$$E(z,t) = E_0 \cos(kz - \omega t)\hat{x}, \qquad B(z,t) = \frac{1}{c} E_0 \cos(kz - \omega t)\hat{y},$$

(a) Ignoring the magnetic force, find the velocity of the particle, as a function of time. (Assume the average velocity is zero.)

(b) Now calculate the resulting magnetic force on the particle.

(c) Show that the (time) average magnetic force is zero.

Problem 4

(a) Using the expression for the magnetic field for a current carrying wire,

$$\vec{B}(\vec{x}) = \frac{I}{c} \oint d\vec{x}' \times \frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3},$$

find the component of the magnetic field along the symmetry (z) axis of a circular current loop of radius a. Show that (origin of the z-axis is in the current plane)

$$B_z(z) = \frac{2\pi I}{c} \frac{a^2}{(a^2 + z^2)^{3/2}}.$$

(b) A long cylindrical solenoid of length $2L$ of circular cross section with radius a consists of N loops of tightly and uniformly wound loops of wire carrying a current I. The origin of coordinates, O, is taken at the center of the solenoid, as shown. In the approximation that the wound loops of wire approximate N separate current loops, show that the magnetic field, B_z^{sol}, of the solenoid along the symmetry axis can be written as

$$B_z^{sol}(z) = \frac{\pi N I}{cL} (\cos\theta_1 + \cos\theta_2)$$

where $\theta_{1,2}$ are the opening angles of the two ends of the solenoid at position z.

(c) Show that very far away from the solenoid, $|z| \gg L$, the expression for the magnetic field on the z-axis is approximately given by

$$B_z^{sol}(z) \approx \frac{2\pi N I}{c} \frac{a^2}{|z|^3}$$

Problem 5

Consider the Dirichlet Green function for the space between the $z = 0$ and $z = L$ planes, for $L > 0$:

$$\nabla_x^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}') = -\frac{4\pi}{\rho}\delta(\rho - \rho')\delta(\phi - \phi')\delta(z - z'),$$

in cylindrical coordinates $\mathbf{x} = (\rho, \phi, z)$ $(\mathbf{x}' = (\rho', \phi', z'))$.

(a)(10 pts) Show that one form of $G(\mathbf{x}, \mathbf{x}')$ is

$$G(\mathbf{x}, \mathbf{x}') = \frac{4}{L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} e^{im(\phi - \phi')}\sin(\frac{n\pi z}{L})\sin(\frac{n\pi z'}{L})I_m(\frac{n\pi}{L}\rho_<))K_m(\frac{n\pi}{L}\rho_>),$$

where I_m and K_m are the modified Bessel functions and $\rho_>$ $(\rho_<)$ is the larger (smaller) of ρ and ρ'.

(b)(10 pts) Using the representation

$$\frac{1}{\rho}\delta(\rho - \rho') = \int_0^{\infty} kJ_m(k\rho)J_m(k\rho')dk,$$

show that $G(\mathbf{x}, \mathbf{x}')$ can also be written as

$$G(\mathbf{x}, \mathbf{x}') = 2\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk\, e^{im(\phi - \phi')}J_m(k\rho)J_m(k\rho')\frac{\sinh(kz_<)\sinh[k(L - z_>)]}{\sinh(kL)},$$

where J_m are the usual Bessel functions and $z_>$ $(z_<)$ is the larger (smaller) of z and z'.

Baylor University Physics Ph.D. Program
Preliminary Exam 2020

Part IV: Statistical Mechanics

Problem 1

A Dicke state is a system of N non-interacting two-level sub-systems. Each two-level sub-system has energy levels ϵ_1 and ϵ_2. This ensemble is in contact with a thermal reservoir with a temperature, T.

(a) What is the Helmholtz free energy, F, for this system in terms of the given parameters and any relevant physical constants?

(b) What is the entropy, S, of the system in terms of the given parameters and any relevant physical constants?

(c) The system temperature is $T = 20K$, $\epsilon_1 = 0$ and $\epsilon_2 = 1.7 meV$. (Note that $1 meV = 1.602 \times 10^{-22} J$). There are $N = 6.02 \times 10^{23}$ sub-systems, as well. What is the specific heat at constant volume, C_V?

Problem 2

(a) Derive the Maxwell relation: $\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$.

(b) Maxwell found that based on his theory of electromagnetic fields, the pressure p of an isotropic radiation field equals one-third of its energy density, $u(T)$, i.e., $p = \frac{1}{3}u(T) = \frac{1}{3}\frac{U(T)}{V}$ in which V is the cavity volume. Use this result, the Maxwell relation in (a), and the fundamental thermodynamic relation $dU = TdS - pdV$ together to prove that $u = \frac{T}{4}\frac{du}{dT}$.

(c) Solve the equation for u and derive Stefan's law for the black body radiation.

Problem 3

Consider an ideal classical gas in a uniform gravitational potential energy $U(z) = mgz$, which measures from the ground level z_0. m is the mass of an individual particle of the gas.

(a) Using the condition of hydrostatic equilibrium for a layer of the gas confined between z and $z + dz$, derive the differential equation for the pressure $\frac{dp(z)}{dz} = -\frac{mg}{k_B T} p(z)$, where k_B is the Boltzmann constant.

(b) For constant temperature, the equation for p is known to result in the isothermal barometric formula $p(z) = p_0 e^{-\frac{mgz}{k_B T}}$. However, air is a poor heat conductor, so to a first approximation, the atmosphere should be regarded as adiabatic. Show in this case that the temperature $T(z)$ decreases linearly with increasing z. Then, derive the corresponding adiabatic barometric formula for $p(z)$. Express your answers in terms of m, g, k_B and $\gamma \equiv \frac{C_p}{C_V}$.

Problem 5

Suppose that a certain system has the partition function $Q(\beta) = \frac{1}{\sqrt{\beta}}$. Evaluate the density of states $g(E)$ for this system using the inverse Laplace transformation explicitly,

$$g(E) = \frac{1}{2\pi i} \int_{\beta'-i\infty}^{\beta'+i\infty} d\beta \; e^{\beta E} Q(\beta) \quad (\beta' > 0).$$

(a) Draw the appropriate closed contour for the integral.

(b) Calculate the integrals along the real axis and show that they are equal to $-\frac{1}{\sqrt{\pi E}}$.

(c) Justify that the contributions from the large and small arcs are zero, and finally evaluate the density of states $g(E)$.

Baylor University Physics Ph.D. Program
Preliminary Exam 2021

Part I: Classical Mechanics

Monday, May 24, 2021, 9:00 a.m.

Problem 1

A two-dimensional object lies in the x, y plane and is described by the moments of inertia I_{xx}, I_{yy} and the product of inertia I_{xy}.

(a) In a coordinate system rotated by an angle ϕ (rotation about the $+z$ axis), find the elements of the new inertia tensor I'_{xx}, I'_{yy} and I'_{xy}.

(b) Find the smallest angle ϕ_0 for which I'_{xy} vanishes. The x' and y' axes are then the principal axes for the object.

(c) If the principal moments are equal and $I_{xy} = 0$ in some coordinate system, show that the moments are equal in any rotated coordinate system.

<u>Problem 2</u>

The polar ice caps contain about 2.3×10^{19} kg of ice. Estimate the change in the length of the day that would be expected if the polar ice caps were to melt. *State clearly all assumptions made in the treatment of the problem.*
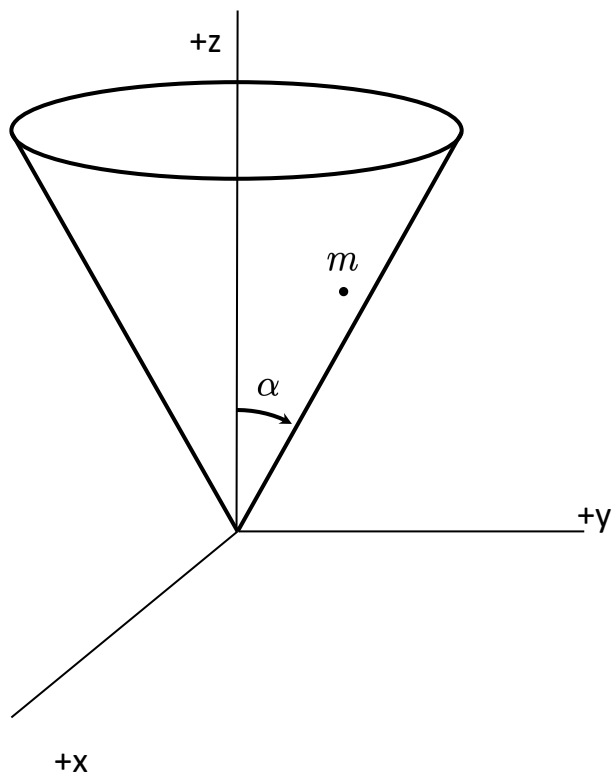
Problem 3

A particle of unit mass moves from infinity along a straight line that, if continued, would allow it to pass a distance $b\sqrt{2}$ from a point P. If the particle is attracted toward P with a force varying as k/r^5, and if the angular momentum about the point P is $\sqrt{k}/b$, show that the trajectory is given by $r = b\coth(\theta/\sqrt{2})$.

Problem 4

A particle of mass m is constrained to move on the surface of a circular cone with opening angle α subject to acceleration due to gravity, g, in the negative z-direction.
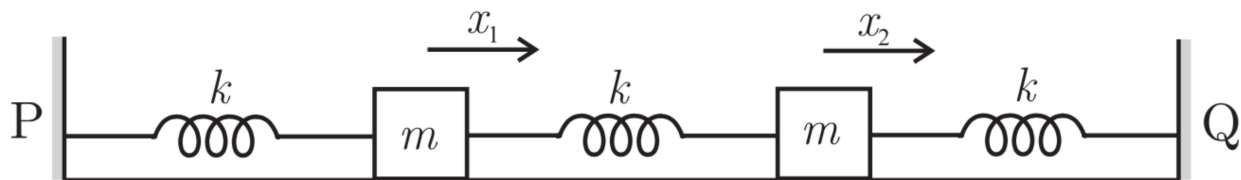

(a) Find the Lagrangian and Lagrange's equations in cylindrical coordinates. Eliminate the constraint by introducing a Lagrange multiplier. Show clearly that one obtains 4 equations in 4 unknowns.

(b) Now eliminate the constraint by adopting spherical coordinates. Find the Lagrangian and Lagrange's equations. Show clearly that one obtains 2 equations in 2 unknowns.

Problem 5

Identical springs (each having force constant k) are connected to two equal masses m as shown below. The masses are constrained to move in one dimension on a frictionless horizontal surface, and the ends of the springs are attached to fixed walls at P and Q.

(a) Find the most general solution for the positions of the masses as functions of time.

(b) What are the normal coordinates?

(c) What are the normal modes?

Baylor University Physics Ph.D. Program
Preliminary Exam 2021

Part II: Quantum Mechanics

Monday, May 24, 2021, 1:30 p.m.

Problem 1

A particle of mass m is in the state

$$\Psi(x,t) = A\exp(-a[(mx^2/\hbar) + it])$$

where A and a are positive, real constants

(a) Find the normalization A.

(b) With what potential energy $V(x)$ does one obtain this wave function $\Psi(x,t)$?

(c) Evaluate the expectation values of x, x^2, p, and p^2.

(d) Find σ_x and σ_p, where $\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ is the standard deviation of position and $\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$ is the standard deviation of momentum. Is their product consistent with the uncertainty principle?

Problem 2

(a) Use the variational principle to estimate the ground-state energy of a particle in the one-dimensional potential

$$V(x) = \infty \quad \text{for} \quad x \le 0,$$
$$= cx \quad \text{for} \quad x > 0.$$

where c is a positive constant. Take $\psi(x) = x\exp(-ax)$ for $x > 0$ (and $\psi(x) = 0$ for $x < 0$) as the non-normalized trial function for $x > 0$, with a being the free parameter to best fit.

A useful integral is (also provided in Shaum's Mathematical Handbook):

$$\int x^n \exp(bx)\,dx = \frac{x^n \exp(bx)}{b} - \frac{n}{b}\int x^{n-1}\exp(bx)\,dx$$
$$= \frac{\exp(bx)}{b}\left(x^n - \frac{nx^n - 1}{b} + \frac{n(n-1)x^{n-2}}{b^2} - \cdots \frac{(-1)^n n!}{b^n}\right),$$

for integer n.

(b) Why is it guaranteed that the estimated ground state energy for any trial wave function will never be less than the actual ground state energy?

<u>Problem 3</u>

An α particle is trapped inside the spherical potential well V of a nucleus, where

$$V(r) = 0 \quad \text{for} \quad r \leq R, \quad \text{(where the attractive strong force dominates)}$$

$$= E_0 \frac{R}{r} \quad \text{for} \quad r > R \quad \text{(with E}_0 > 0, \text{and denoting where the repulsive Coulomb force dominates)}$$

The α particle begins in a bound state with energy $Q = E_0/10$. In general, a particle's tunneling probability P through a potential barrier is proportional to an inverse exponential, which depends on the barrier width and the square root of the barrier height ΔE:

$$P = \exp[-\frac{2\sqrt{2m}}{\hbar} \int_R^{R'} \sqrt{\Delta E(r)} dr].$$

where m is the particle mass, $\Delta E = V(r) - Q$ is the barrier height at a given radius r, and R' is the distance at which $V(r) = Q$, and the particle can "escape" through the barrier.

Find the tunneling probability for this α particle, in terms of m, $\hbar$, E_0, and R. You may need to use the integral:

$$\int \sqrt{\frac{a}{x} - 1}\, dx = x\sqrt{\frac{a}{x} - 1} - a \tan^{-1} \sqrt{\frac{a}{x} - 1}.$$

Problem 4

Consider a spin-1/2 particle with a magnetic moment $\mu = e\hbar/(2m_e c)$ moving in a constant external magnetic field $\vec{B} = B\hat{z}$, where B is a constant, and $\hat{z}$ is a unit vector along the z-axis. Then, the corresponding Hamiltonian is give by

$$H = -\vec{\mu} \cdot \vec{B} = -\frac{e}{m_e c} \vec{S} \cdot \vec{B} = \omega S_z$$

where $\omega \equiv |e| B/(m_e c)$.

(a) From the Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} U(t) = HU(t),$$

find the time-evolution operator $U(t)$, by assuming that at the initial time $t = 0$, we have $U(0) = 1$.

(b) Assuming that the particle is initially in the state $|\alpha\rangle = \sin\theta|+\rangle + \cos\theta|-\rangle$, where θ is a real constant, and $|\pm\rangle$ denotes the eigenstates of S_z with $S_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$, find the state of the particle at time t.

(c) Find the probabilities of the particle to be in the $|S_x, \pm\rangle$ states, where

$$|S_x, \pm\rangle = \frac{1}{\sqrt{2}}(|+\rangle \pm |-\rangle).$$

Problem 5

Consider the energy spectrum of a system whose Hamiltonian is,

$$H = H^0 + H^1 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 + ax^3 + bx^4,$$

where a and b are small constants. H is an anharmonic oscillator which reduces to a harmonic oscillator if $a = b = 0$. Accordingly, use perturbation theory with

$$H^1 = ax^3 + bx^4,$$

to find the perturbed energy eigenvalues to the first order. Let $x_0^2 = \frac{\hbar}{m\omega}$ and $\xi = x/x_0$. You may wish to use the recursion relation,

$$\xi\psi_n(\xi) = \sqrt{n/2}\,\psi_{n-1}(\xi) + \sqrt{(n+1)/2}\,\psi_{n+1}(\xi),$$

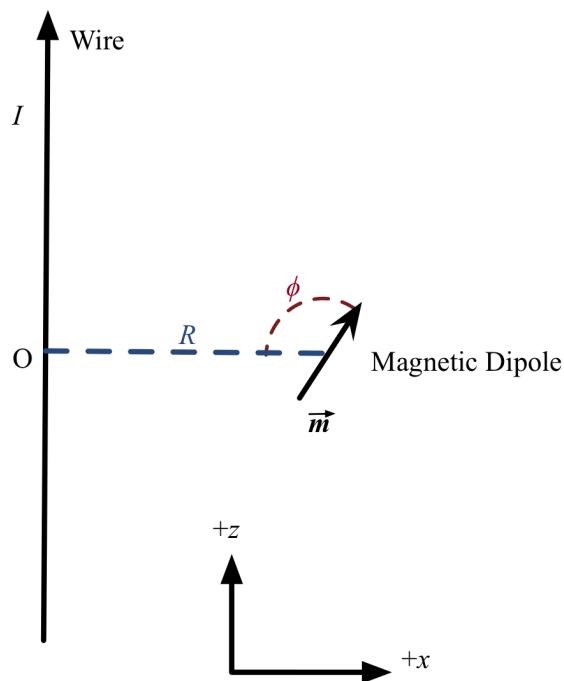where ψ_n is the nth energy eigenstate of the harmonic oscillator.

Baylor University Physics Ph.D. Program
Preliminary Exam 2021

Part III: Electricity & Magnetism

Tuesday, May 25, 2021, 9:00 a.m.

<u>Problem 1</u>
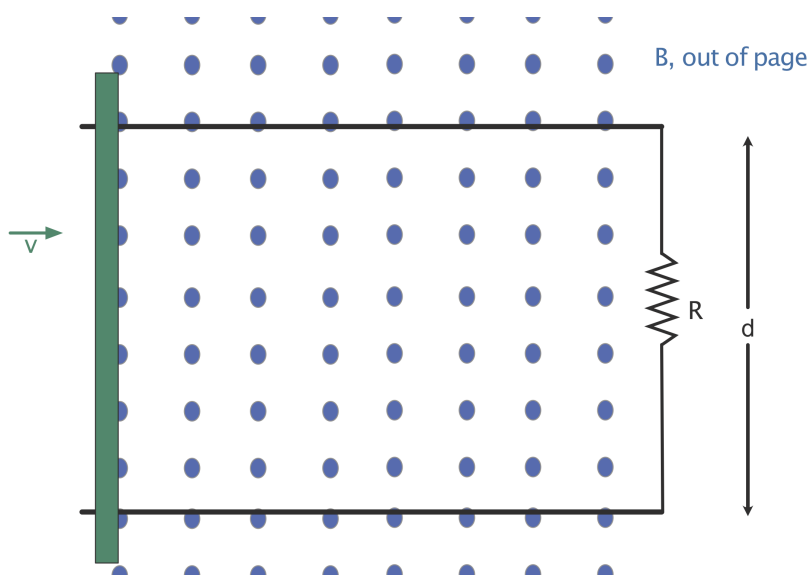
Consider a small magnetic pole with a magnetic moment, $\vec{m}$, a distance ρ away from a very long, straight, and thin wire caring a current, I. The magnetic pole is located on the x-axis relative to the origin, O, as shown. Find the components of the force, $\vec{F}$, on the magnetic dipole.
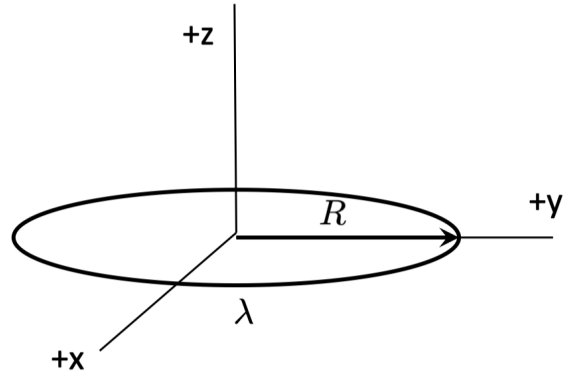
Problem 2

The $d = 9.0$ cm wide slide wire shown in the Figure below is pushed toward the $R = 35.2\,\Omega$ resistor at a constant velocity, $\vec{v} = (3.2\,\mathrm{m\,s^{-1}})\hat{x}$. The rail on which this is pushed is metal and has a negligible resistance (beyond the discrete resistor indicated). The uniform magnetic field strength is $|B| = 25.00$ T and is pointed out of the page ($\hat{z}$). The slide wire has a negligible resistance and there is no friction between the slide wire and the rail.
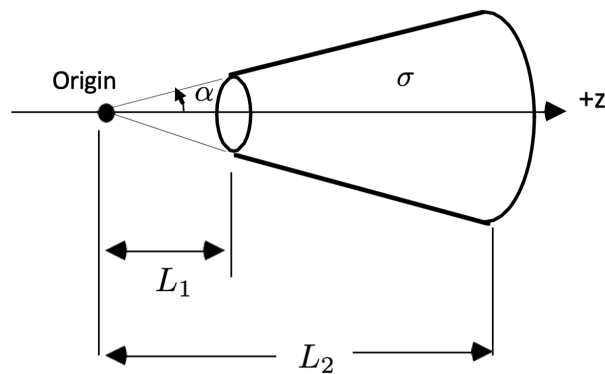

(a) What is the net force, $\vec{F}_{net}$, on the slide wire?

(b) With how much force, $\vec{F}_{push}$, do you need to push this to maintain a steady speed, v?

(c) What is the direction and magnitude of the induced current, i?

(d) How much power, P, is dissipated in the resistor?

Problem 3

(a) A circular loop of radius R with a uniform linear charge density λ is oriented in the x-y-plane. Find the electric field, $\vec{E}$, along the z axis.
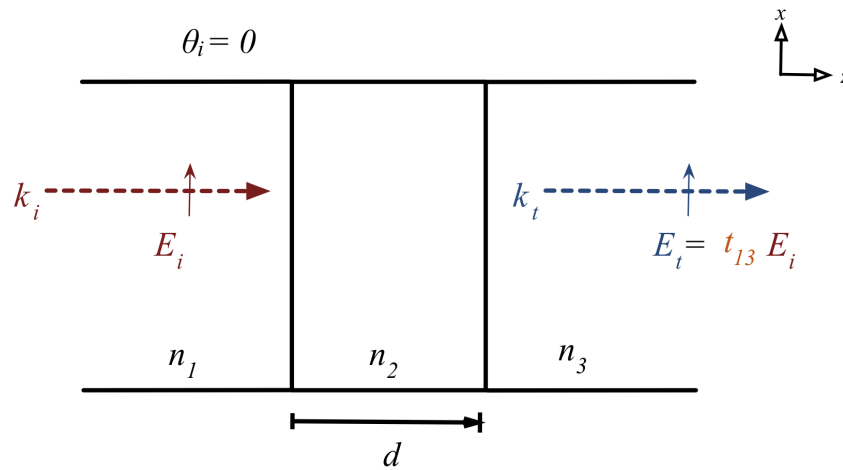

(b) A truncated circular cone with an opening angle α has its symmetric axis oriented along z-axis. The origin is taken at the theoretical tip of the cone. However, the end of the cone is shaved off a distance L_1 along the z-axis from the origin and the distance from the origin to the top of the cone along z is L_2. The cone has a uniform charge density σ smeared on its surface. Find $\vec{E}$ at the tip of the cone.
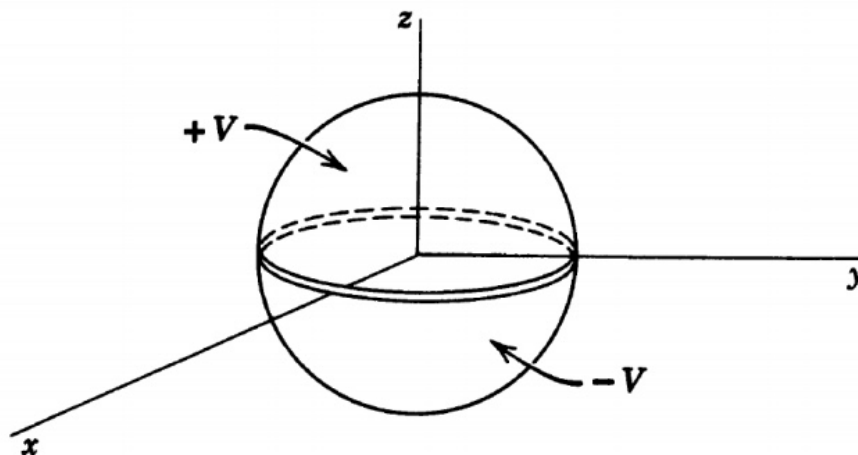
Problem 4

Light of frequency ν, with its electric field E_i, originates in medium n_1, transmits through a medium n_2 of thickness d, and into medium n_3. The reflection coefficient $r_{ij} = \frac{n_i - n_j}{n_i + n_j}$ is defined for reflection in medium i off medium j. The transmission coefficient $t_{ij} = \frac{2n_i}{n_i + n_j}$ is likewise defined for transmission from medium i into medium j.

Find the field transmission coefficient, $\tilde{t}_{13}(\nu)$, for light from medium 1 to medium 3 through medium 2 if this light is at normal incidence on each boundary. Assume all three media are linear, homogeneous, and have $\mu_1 = \mu_2 = \mu_3 = \mu_0$.

Problem 5

A conducting sphere of radius R is centered at the origin. It is made of two hemispherical shelves separated by a small insulation ring. The hemispheres are kept at different potentials.

(a) Construct the Green's function, $G(\vec{r},\vec{r'})$, for this problem.

(b) What is the potential, $\Phi(x = 0, y = 0, z)$, along the z-axis?

Baylor University Physics Ph.D. Program
Preliminary Exam 2021

Part IV: Statistical Mechanics
Tuesday, May 25, 2021, 1:30 p.m.

Problem 1

A $1V$ battery is connected to a $2\,\Omega$ resistor for 100 seconds. The circuit, thermally isolated, has an initial temperature of $300\,K$. The heat capacity of the resistor is $0.24\,J/K$ and is constant over a wide range of temperatures.

What is the entropy change of the system?

Problem 2

In a certain thermodynamic system, the partition function Z has a closed form given by:

$$Z = \exp(\alpha k_B^4 T^4 V)$$

where α is a constant, and k_B is the Boltzmann constant. Also, T and V are temperature and volume, respectively. Determine:

(a) The Pressure

(b) The Entropy

(c) The Energy of the system.

Problem 3

Consider a three-dimensional non-relativistic Fermi gas consisting of N spinless particles with mass m in a volume V at temperature T. Do the following:

(a) Express the Fermi wave-vector k_F in terms of N and V by explicit evaluation of k or the momentum integral.

(b) Next, denoting the Fermi energy by $\epsilon_F (= k_F^2/(2m))$, assume that the temperature dependence of the chemical potential μ is given by:
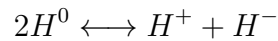
$$\mu = \epsilon_F - \frac{bT^2}{\epsilon_F}$$

where b is a constant. Using this expression, determine the entropy S via an appropriate Maxwell relation. Note: for the Helmholtz free energy, $F(T, V, N) = U(S, V, N) - TS$, the differential form is $dF = -SdT - pdV + \mu dN$. You may further assume that the entropy S vanishes at $N = 0$.

(c) Using the entropy S obtained in part (b), calculate $C_{V,N}$, the specific heat at constant N, V.

Problem 4

In the solar photosphere, hydrogen can exist as neutral atom H^0, positively charged ion H^+ (or proton), and negatively charged ion H^-, with the electron occupation numbers $n = 1, 0, 2$, respectively. The equation for reaction can be written as

$$2H^0 \longleftrightarrow H^+ + H^-$$

Note that the H^0 level is doubly degenerate due to electron spin. The two ionized states are non-degenerate. Also, let the ionization energy of H^0 be $E(n = 1) = -\Delta$, relative to $E(n = 0) = 0$ for the zero occupancy. The ionization energy of H^- is ϵ, that is the energy to extract an eletron from H^-. Thus, $E(n = 2) = -\Delta - \epsilon$.

(a) Draw the energy level scheme for the three possible occupancies and find the grand partition function.

(b) Assuming that $\langle n \rangle \approx 1$, show that the chemical potential $\mu = -\frac{(\Delta + \epsilon)}{2}$.

(c) Further, show that $\langle \Delta n^2 \rangle = \left(1 + e^{\beta \frac{(\Delta - \epsilon)}{2}}\right)^{-1}$.

Problem 5

For non-relativistic bosons in 3D, the number of particles can be obtained in dimensionless form,

$$N \propto \int_0^\infty \frac{x^{1/2}}{e^x - 1} dx$$

as temperature approaches zero and fugacity $z = 1$.

(a) Explicitly evaluate the integral to show that

$$\int_0^\infty \frac{x^{1/2}}{e^x - 1} dx = \Gamma\left(\frac{3}{2}\right) \zeta\left(\frac{3}{2}\right)$$

where Gamma function $\Gamma(\nu) = \int_0^\infty x^{\nu-1} e^{-x} dx$ and Riemann zeta function $\zeta(s) = \sum_{n=1}^\infty n^{-s}$.

(b) Test $\zeta(s) = \sum_{n=1}^\infty n^{-s}$ for convergence.

Classical Mechanics
Preliminary Exam, Spring 2022

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Grading: There are **5** questions on this exam, which is scored out of **100 pts**. Unless otherwise noted, all parts are required.

Identification Number: _____

Problem	1	2	3	4	5	Total
Points:	20	20	20	20	20	100
Score:						

Classical Mechanics

1. 20 points Consider a square plate of side a and mass m in a coordinate system with the origin at one corner and x and y along the edges of the plate.

   (a) Find the components of the inertia tensor for the plate in this coordinate system. Use symmetry arguments to simplify the calculations.

   (b) Determine the angular momentum when the plate above is rotated about the diagonal $x = y$.

   (c) Find the kinetic energy of the rotating plate in (b)

2. For the Lagrangian:
$$L = \frac{m}{2}\left[\dot{q}_1^2\left(1 + \sin^2 q_2\right) + \dot{q}_2^2\right] - \frac{k}{3}q_2^3$$

   where k and m are constants.

   (a) 8 points Find the equations of motion. (Don't try to solve the equations.)

   (b) 4 points Are there any ignorable (i.e. cyclic) coordinates? If so, what are the corresponding conserved momenta?

   (c) 8 points Find the energy of the system. Is it conserved? Why?

3. 20 points In a linear triatomic molecule, two atoms of mass m are symmetrically located on each side of an atom of mass M (see figure). All three atoms are on one straight line and are in equilibrium positions. Only vibrations along the line of the molecule are taken into the consideration. The interatomic potential is approximated by two springs joining the three atoms. The force constant of the spring is k and the equilibrium length is b.

   (a) Find the eigenfrequencies of the system.

   (b) Find the eigenmodes and describe the motion of the atoms corresponding to each eigenfrequency.

4. 20 points Consider the orbit equation $(r = 1/u)$
$$\frac{d^2u}{d\phi^2} + u = -\frac{\mu}{\ell^2}\frac{d}{du}V\left(\frac{1}{u}\right),$$

   where μ is the reduced mass, ℓ is the relative angular momentum and ϕ is the orbital angle. The gravitational potential $V\left(\frac{1}{u}\right)$ is given by
$$V\left(\frac{1}{u}\right) = \frac{\alpha}{u^k},$$

where $\alpha k > 0$. By perturbing the orbit equation, show that the total angular change in going from r_{max} back to r_{max} (or r_{min} to r_{min}) for a nearly circular orbit, $u \approx u_0 + \epsilon$ ($|\epsilon| << u_0$), is

$$\Delta\phi = \frac{2\pi}{\sqrt{k+2}}.$$

5. 20 points A small meteoroid of mass m is in a circular orbit of radius R around a very heavy stellar object of mass M. Let Newton's gravitational constant be G.

(a) Find the speed v of the meteoroid, its angular momentum L, and its total mechanical energy E.

(b) Suppose the stellar object suddenly explodes in a spherically symmetric way without gaining any momentum, and loses part of its mass to a region far away from the system, becoming a neutron star of mass $M_f < M$. Assume that the explosion happens very fast compared to the motion of the meteoroid, and the motion of the meteoroid is not changed during the explosion. After the explosion, the meteoroid moves in the gravitational field of the newly born neutron star.

   i. What is the angular momentum of the meteoroid before and after the explosion?

   ii. What is the total energy of the meteoroid before and after the explosion?

   iii. Find the minimum value of M_f such that the meteoroid's orbit is still bound to the neutron star.

(c) For the case where $M_f = \frac{3}{4}M$, find the largest distance between neutron star and the meteoroid (r_{max}), and the speed of the meteoroid at that location $(v(r_{max}))$.

Quantum Mechanics
Preliminary Exam, Spring 2022

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Identification Number: _____

Problem	1	2	3	4	5	Total
Points:	20	20	20	20	20	100
Score:						

Quantum Mechanics

1. 20 points A particle moving on a sphere is described by the wavefunction (in spherical coordinates), $\psi = C \sin\theta \sin\varphi$, where C is a constant.

   (a) What is the expectation value of L_z for this state?

   (b) What are the possible outcomes of the measurements of L_z, and what are their probabilities?

   (c) What is the expectation value of L_x ?

   (d) What is the expectation value of L^2 ?

   (e) Find the normalization constant C, and the expectation values of $\cos^2\theta$.

2. 20 points Consider a particle of mass m moving in an infinitely deep potential well of width a: $V = 0$ for $0 < x < a$ and $V(x) = \infty$ for $x < 0$ and $x > a$.

   (a) Find the energy levels E_n and corresponding wave-functions ψ_n for all stationary states $n = 1, 2, \ldots$.

   (b) Consider a particle initially in the ground state ψ_1 . One of the walls is then moved with velocity v until the width of the well doubles: $V(x) = 0$ for $0 < x < 2a$. What is the probability that the particle will be found in the first excited state $(\tilde\psi_2)$ in the new potential if the wall moves infinitely fast $(v \to \infty)$.

   (c) Again, for a particle initially in the state ψ_1 , what is the probability to find it in the first excited state of the new potential if the wall moves adiabatically slowly $(v \to 0)$.

   (d) What is the condition for the velocity v of the wall for the adiabatic approximation to be applicable: $v \ll$ ____ ?

3. 20 points The normalized 1-dimensional wavefunction of a particle in momentum space is:

$$\phi(p) = \sqrt{\frac{1}{\delta}} \exp\left[-\frac{|p - a|}{\delta}\right]$$

   (a) Express the wavefunction in position space.

   (b) Show that Heisenberg's Uncertainty Principle, $\Delta x \Delta p \geq \frac{\hbar}{2}$, is obeyed by this particle.

4. (a) 6 points Show that if a system is in an eigenstate of J_z, the mean value of the operators J_x and J_y vanishes.

   (b) 7 points Show that for a system in the eigenstate $|j, m_j\rangle$ of the operator J_z , the mean value of the component of angular momentum along a direction $\hat{z}'$, which makes an angle θ with the z-axis, is equal to $m_j \cos\theta$.

   (c) 7 points Since the components of the angular momentum operator do not commute, their simultaneous measurement is not possible. Show that in a state $|j, m_j\rangle$, the greatest accuracy of measurement of the components J_x and J_y is obtained when $|m_j| = j$.

Quantum Mechanics

5. 20 points Consider an electron at rest in a uniform magnetic field, $\vec{B} = B_z\hat{z}$, for which the Hamiltonian is:

$$\hat{H}_0 = \frac{eB_z}{m}\hat{S}_z$$

Assume $B_z > 0$ for this entire problem.

(a) Find the unperturbed eigenspinors and corresponding eigenvalues of $\hat{H}_0$

(b) Consider a perturbation in the form of a uniform electric field in the x direction.

$$\hat{H}' = \frac{eB_x}{m}\hat{S}_x$$

Using perturbation theory, calculate the approximate ground state energy correct to second order.

(c) Now find the exact ground state energy and eigenspinor for the full Hamiltonian (i.e. $\hat{H} = \hat{H}_0 + \hat{H}'$). The eigenspinor need not be normalized. Compare your answer with the result for (b) for $|B_x| \ll |B_z|$.

Electricity and Magnetism
Preliminary Exam, Spring 2022

Identification Number: _____

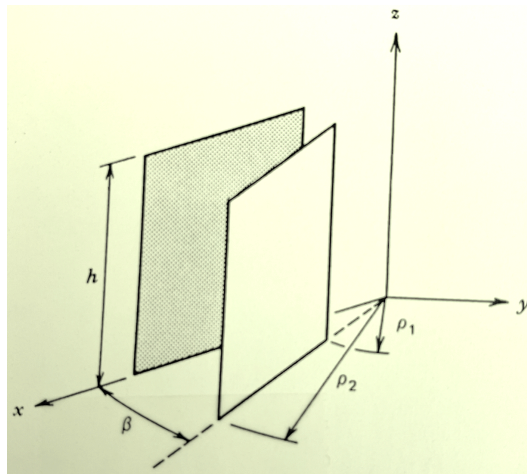| Problem	1	2	3	4	5	Total
Points:	20	20	20	20	20	100
Score:						

Electricity and Magnetism

1. 20 points A conducting square wire loop with sides of length L has a mass of M and a total resistance R. It lies flat in the $x - y$ plane, centered at $(0,0)$ with the x and y axes bisecting its sides. A uniform magnetic field of strength B passes through the plane in the $\hat{z}$ direction. The square loop rotates freely around the x-axis, with an initial angular velocity ω_0, and an initial angle $\theta = 0°$ above the horizontal.

   (a) What is the initial rotational kinetic energy of the square?

   (b) What is the power dissipated by the induced current in the square loop as a function of $\omega(t)$ and $\theta(t)$?

   (c) Find an expression for the kinetic energy of the square loop as a function of time, assuming that the square makes many full rotations in the time it takes for its kinetic energy to change by a small amount?

2. 20 points Calculate the electrostatic potential, Φ, correct to the dipole approximation for the following charge distribution.

$$\rho(r, \theta, \phi) = \rho_0 e^{-\frac{r}{a}} \sin\theta \sin\phi$$

3. 20 points Consider two large metallic plates forming a wedge capacitor as shown above. The plate at $\varphi = 0$ is held at zero volts while the plate at $\varphi = \beta$ is held at V volts. Neglect fringing effects.
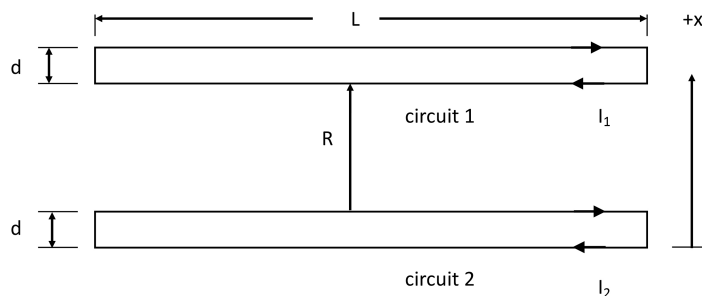

   (a) Write down the differential equation satisfied by the potential inside the capacitor and determine the potential.

   (b) Determine the charge density and the total charge on the plates.

4. 20 points (a) Calculate the mutual inductance per length, $M_{12}(R)/L$, of two long identically shaped flat rectangular circuits with parallel sides, L , which are much longer than the end sides, d. Neglect edge effects. The distance R is measured between the near sides of circuits 2 and 1, as shown, and the x-coordinate is measured

from the bottom of circuit 2. Show that:

$$M_{12}(R)/L = -\frac{2}{c^2} \ln \frac{(R+d)^2}{R(R+2d)} \quad \text{(Gaussian)}$$

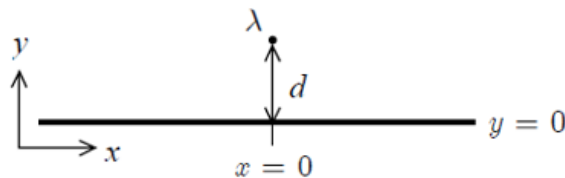$$M_{12}(R)/L = -\frac{\mu_0}{2\pi} \ln \frac{(R+d)^2}{R(R+2d)}. \quad \text{(SI)}$$

(b) Using the connection between mutual inductance and force, find the force of circuit 2 on 1, F_{12}. Alternatively, calculate the force in the x-direction using the Lorentz expression involving magnetic field and currents. Show that:

$$(F_{12})_x = \frac{4I_1 I_2}{c^2} \frac{d^2}{R(R+2d)(R+d)} \quad \text{(Gaussian)}$$

$$(F_{12})_x = \frac{\mu_0 I_1 I_2}{\pi} \frac{d^2}{R(R+2d)(R+d)}. \quad \text{(SI)}$$

5. $\boxed{20 \text{ points}}$ A line-charge with charge per length λ is oriented in the z-direction and placed a distance d above an infinite grounded conducting plane, the x-z plane located at $y = 0$.

(a) Find the electrostatic potential in the region $y > 0$.

(b) What is the 2-D Greens function, $G(x, y; x', y')$, for this geometry?

Statistical Mechanics
Preliminary Exam, Spring 2022

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Score:						

Statistical Mechanics

1. 20 points The Riemann zeta function $\zeta(s) = \sum\limits_{n=1}^{\infty} n^{-s}$ is found in many applications in statistical thermodynamics, for example in blackbody radiation as well as in the Sommerfeld expansion for a strongly degenerate electron gas. One can obtain a particular integer value by taking advantage of a polynomial function expanded in a Fourier expansion.

   By explicitly expanding $f(x) = x^2$ in a Fourier series, evaluate $\zeta(2) = \frac{\pi^2}{6}$.

2. Consider a perfect Bose gas of spin $S = 1$ that is confined in a two-dimensional plane with area A.

   (a) 7 points Show that the Bose gas does not undergo Bose-Einstein condensation by explicitly evaluating the integral

   $$N = \int g(\varepsilon) f(\varepsilon) d\varepsilon$$

   where $g(\varepsilon)$, $f(\varepsilon)$, and ε are density of states, the occupancy functions, and the single particle energy, respectively.

   (b) 7 points Suppose that a central potential of the form $\phi(r) = ar^2$, where r is measured from some point in the $2D$ plane, is turned on. Show that density of states is now $g(\varepsilon) = \frac{\varepsilon}{(\hbar\omega)^2}$. Hint: Consider the potential energy $\phi(r) = ar^2$ as a $\phi(x, y) = ax^2 + ay^2$ in two-dimensions.

   (c) 6 points Assuming the potential in Part b, show that Bose-Einstein condensation can take place.

3. 20 points A body of constant heat capacity C_p and initial temperature T_1 is placed in contact with an infinite reservoir at temperature T_2. Equilibrium is established at constant pressure.

   (a) Find the change of the entropy of the body.

   (b) Find the change of the entropy of the reservoir.

   (c) Show that the total entropy increases regardless of the conditions $T_1 > T_2$ or $T_1 < T_2$.

4. 20 points N cubes of mass, m, and dimensions $w \times w \times w$ are laid one on top of another to make a wall that is $N \times w$ high. The wall is knocked over and all cubes come to rest on the floor in a room at temperature T. What is the change of entropy in the Universe for $N > 1$?

5. $\boxed{\text{20 points}}$ An assembly of N particles of spin-$\frac{1}{2}$ are lined up on a straight line. Only nearest neighbors interact. When the spins of the neighbors are both up or both down, their interaction energy is J. When one is up and one is down, their interaction energy is $-J$. In quantum-mechanical language, the energy is $J\sigma_z^i\sigma_z^j$ between a neighboring pair i and j. Let N_p be the number of the nearest neighbor pairs with parallel spins and N_A be the number with anti-parallel spins. What is the partition function Z of the assembly at temperature T?

Classical Mechanics
Preliminary Exam, Spring 2023

1. The Koch snowflake is a fractal shape, as shown in Figure 1 below. This snowflake is flat in the x-y plane, with a uniform mass density, and total mass M. The snowflake is centered at the origin and has diameter $D = 3$, as can be seen from its length along the y-axis.

   The Koch snowflake can be divided into seven smaller snowflakes of identical shapes. The six small exterior snowflakes have diameter $d = D/3 = 1$, while the medium-sized central snowflake has diameter $d' = D/\sqrt{3} = \sqrt{3}$.

   (a) Derive an expression for the rotational moment of inertia I_z of the whole snowflake about the z-axis (out of the page), passing through the center of the snowflake, in terms of the moments of inertia of the seven smaller snowflakes, their masses, and/or their positions within the large snowflake.

   (b) Use this relation to solve for the total moment of inertia of the whole large snowflake I_z in terms of M and D.
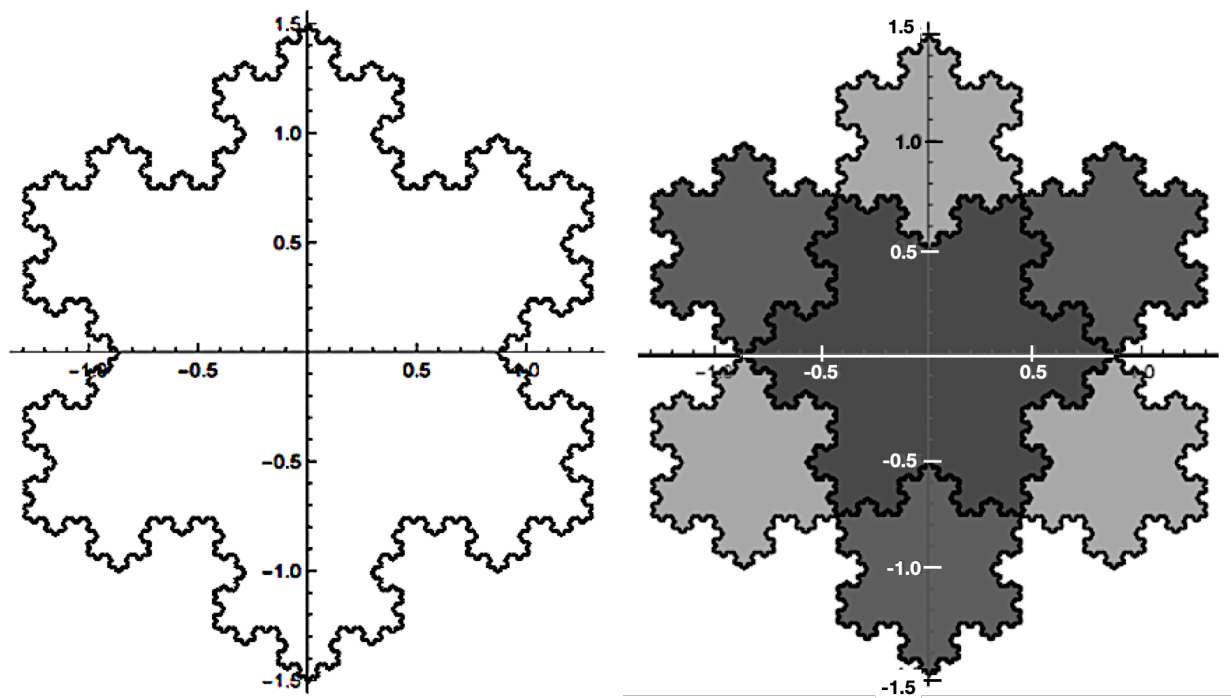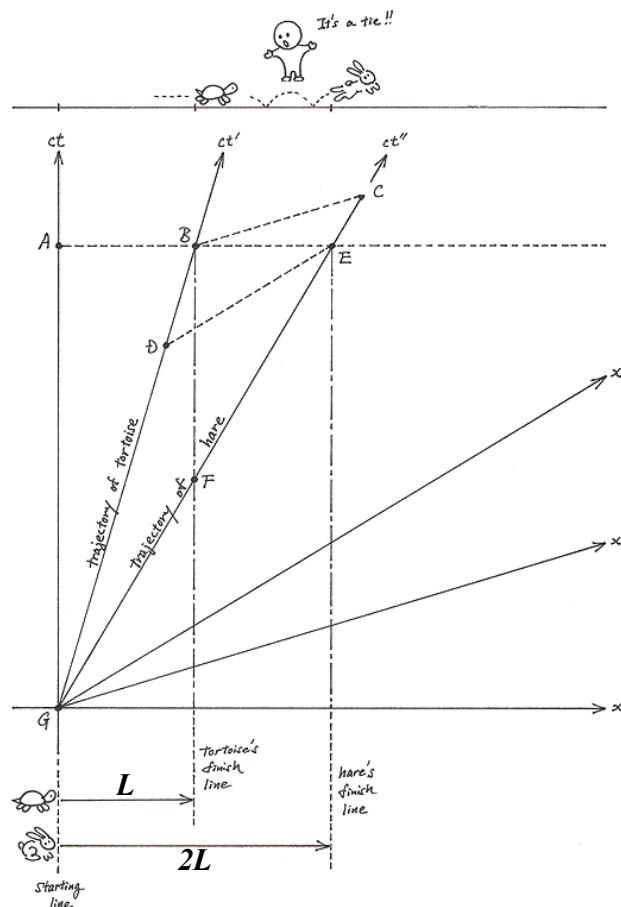

Figure 1: The Koch snowflake is the area enclosed by the Koch fractal curve (left). The snowflake can be divided into seven self-similar snowflakes (right). Figure by Mark McClure.

2. A hare and a tortoise decide to race. They start from the same point and race in the same direction, but the (generous) hare decides to let the tortoise race only half the distance, as shown below. The race takes place. The referee, who is at rest with respect to the ground, sees both animals cross their respective finish lines at the *same time*. The trajectory of the hare and the tortoise are shown on the space-time diagram.

   In the rest frame of the referee, the distance from start to finish for the tortoise is L and the distance from start to finish for the hare is $2L$. According to the referee, the race takes place for a time T. Let c denote the speed of light.

   (a) In the tortoise's frame, how long did it take the tortoise to go from start to finish?

   (b) In the tortoise's frame, when the tortoise crosses its finish line, what is the distance from the tortoise to the starting line?

   (c) In the tortoise's frame, when the tortoise crosses its finish line, what is the distance from the tortoise to the hare?

   (d) In the respective frames of the hare and the tortoise, which animal wins the race? Do they agree or disagree? If they disagree, explain why it is not a contradiction. (You may use the space-time diagram to make your explanation easier.)
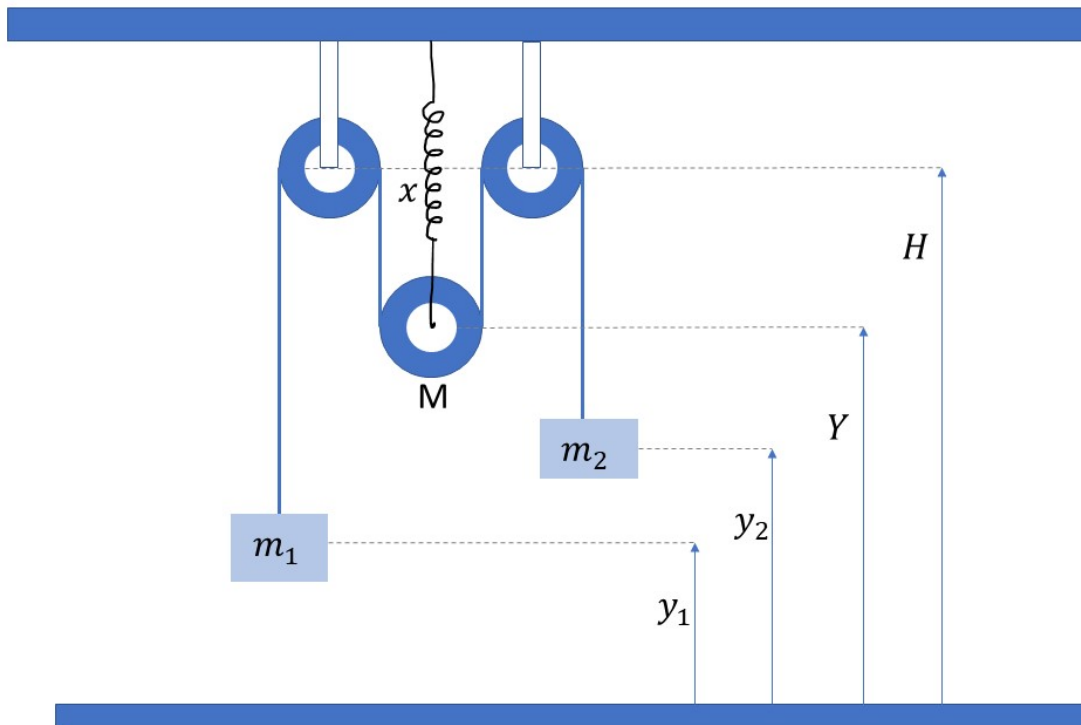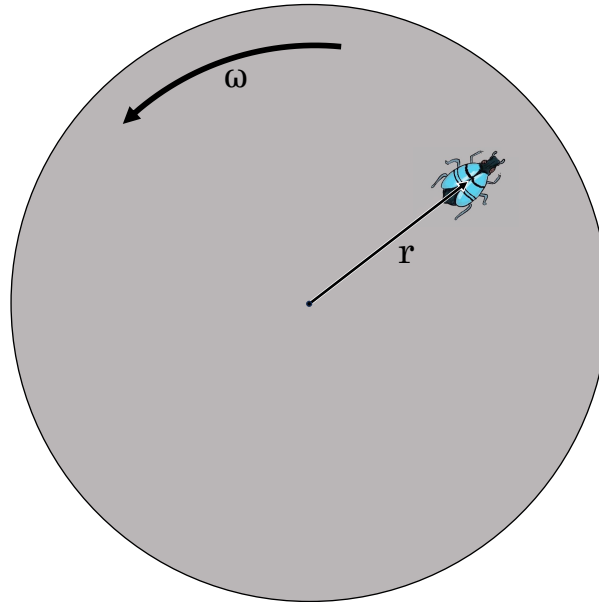
3. The figure below shows a set of three pulleys supporting three weights. The two outside pulleys are fixed to the ceiling at height H above the ground, and the third central pulley is free to move up and down, but is connected to the ceiling by a massless spring with spring constant k, length x, and natural length x_0. Two weights with masses m_1 and m_2 hang from the outer two pulleys, while a weight of mass M is attached to the middle pulley. Assume that the pulleys and the connecting rope have negligible mass and that the rope cannot stretch. Use g for the gravitational constant of acceleration.

   (a) Out of the four unknown distances (y_1, y_2, Y, and x), how many independent degrees of freedom remain after applying the constraints of the rope and pulley system?

   (b) Write the complete set of Euler-Lagrange equations which describe the accelerations of m_1, m_2, and M, based on the number of degrees of freedom in part (a).

   (c) Determine the frequency of small oscillations.

4. A horizontal disk is rotating at a constant angular speed ω. A bug crawls outward on the disk along a radial line such that bug's distance from the center is $r = bt^2$, where b is a constant. Find the acceleration vector of the bug and the magnitude of the acceleration in terms of ω, b, and t.

5. A particle with mass m starts from rest at $x_0 > 0$ and experiences a force $F(x) = -k/x^3$ along the x axis, where k is a positive constant. Show that the time taken to reach the origin is $T_0 = \sqrt{m/k}\ x_0^2$.

Quantum Mechanics
Preliminary Exam, Spring 2023

Instructions: Show all work on the pages provided. These pages have a unique QR code used to sort the problems for grading.

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Grading: There are **5** questions on this exam. Each problem is worth 20 points.

1. 20 points Let P and Q denote two different beams of atoms moving in the $+y$ direction. The atoms in both beams have zero orbital angular momentum $(l = 0)$ and spin quantum number $s = \frac{1}{2}$. The atoms in P have spins that are completely aligned along the $+x$ axis, while the atoms in Q have spins that are completely *unpolarized*.

   (a) Let $|+\rangle_z$ denote the eigenfunction of an atom with zero orbital angular momentum and $s = +\frac{1}{2}$ along the $+z$ axis. Let $|-\rangle_z$ denote the eigenfunction of an atom with zero orbital angular momentum and $s = -\frac{1}{2}$ along the $+z$ axis. What is the spin state function of P in terms of $|+\rangle_z$ and $|-\rangle_z$? Explain your answer.

   (b) If the two beams are passed separately through a Stern-Gerlach apparatus with its (inhomogenous) magnetic field along the $+z$ axis, is there any difference between the emerging beams in the two cases? Explain your answer.

   (c) Explain how the difference between beams P and Q could be detected experimentally.

2. $\boxed{20 \text{ points}}$ A one-dimensional asymmetric infinite potential well is shown in the figure.

(a) Show that the (unnormalized) wavefunction

$$
\psi(z) =
\begin{cases}
0, & z < -b \\
-\,C_n \sin\left[\beta_n(z + b)\right], & -b \leq z < 0 \\
(1/C_n) \sin\left[\alpha_n(z - a)\right], & 0 \leq z \leq a \\
0, & a < z
\end{cases}
$$

is a solution to Schrödinger's equation for $E > -V_1$, where α_n and β_n are defined in terms of the eigenenergies E_n and other constants.

(b) Find $C_n{}^2$ in terms of α_n, β_n, and other constants.

(c) Show that the energy-eigenvalue equation for $E > -V_1$ is

$$
\alpha_n \tan(\beta_n b) = -\beta_n \tan(\alpha_n a).
$$

3. 20 points Consider the three-dimensional motion of a particle, and denote its position and linear momentum by $\vec{x}$ and $\vec{p}$, respectively, where $\vec{x} \equiv (x, y, z) \equiv (x_1, x_2, x_3)$ and $\vec{p} \equiv (p_x, p_y, p_z) \equiv (p_1, p_2, p_3)$.

(a) The classical Poisson bracket of any two continuous functions of generalized coordinates $f(\vec{p}, \vec{q})$ and $g(\vec{p}, \vec{q})$ is defined to be

$$\{f, g\}_{\text{classical}} \equiv \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right).$$

Calculate the classical Poisson brackets

$$\{x_i, F(\vec{p})\}_{\text{classical}} \quad \text{and} \quad \{p_i, G(\vec{x})\}_{\text{classical}},$$

where F and G are analytical functions of their indicated arguments.

(b) What are the corresponding relations in quantum mechanics?

(c) Evaluate the commutator $\left[x_j, \exp\left(\frac{i\vec{a} \cdot \vec{p}}{\hbar} \right) \right]$, where $j = 1, 2,$ or 3 and $\vec{a} \equiv (a_1, a_2, a_3)$ is a constant vector.

(d) Prove that $|\vec{w}\rangle \equiv \exp\left(\frac{i\vec{a} \cdot \vec{p}}{\hbar} \right) |\vec{x}\rangle$ is an eigenstate of x_j ($j = 1, 2,$ or 3) and find the corresponding eigenvalue.

4. 20 points Consider the one-dimensional attractive potential $V(x) = -\alpha\delta(x)$ with $\alpha > 0$.

   (a) Assume a Gaussian trial wavefunction for the ground state:

   $$\phi_0(x) = \frac{\sqrt{q}}{\pi^{1/4}}e^{-q^2x^2/2}$$

   where q is a free parameter. Using this wavefunction, calculate $\langle V \rangle$ and $\langle T \rangle$, the expectation values of the potential energy and kinetic energy, respectively.

   (b) Sketch $\langle V \rangle$, $\langle T \rangle$, and the total energy E as a function of q. Find q_{min}, the value of q that provides the lowest total energy E_{min}.

   (c) Show that $E_{min} = -C\frac{m\alpha^2}{\hbar^2}$, and find the constant C.

   (d) The wavefunction

   $$\psi_0(x) = \frac{\sqrt{m\alpha}}{\hbar}e^{-m\alpha|x|/\hbar^2}$$

   satisfies Schrödinger's equation for $V(x)$ and represents an actual ground-state solution. What is the corresponding ground-state energy E_0? Compare E_{min}, your estimate of the ground-state energy, to the actual ground-state energy E_0.

5. 20 points Particles of mass m are incident on a spherically symmetric potential

$$V(r) = \frac{\beta}{r} \exp(-\gamma r)$$

where β and γ are constants.

(a) Show that, in the Born approximation, the differential scattering cross section for the scattering vector $\vec{K} \equiv \vec{k}_i - \vec{k}_f$ is given by

$$\frac{d\sigma}{d\Omega} = \left[\frac{2m\beta}{\hbar^2(K^2 + \gamma^2)}\right]^2.$$

(b) Use this result to derive the Rutherford formula for the scattering of α-particles, namely, that for α-particles of energy E incident on nuclei of atomic number Z, the differential scattering cross section for scattering at an angle θ to the incident direction is (in SI units)

$$\frac{d\sigma}{d\Omega} = \left[\frac{Ze^2}{8\pi\epsilon_0 E \sin^2(\theta/2)}\right]^2.$$

Electricity and Magnetism
Preliminary Exam, Spring 2023

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Grading: There are **5** questions on this exam. Each problem is worth 20 points.

1. $\boxed{20 \text{ points}}$ Consider a spherical charge distribution of radius a with uniform charge density ρ_0. Calculate the energy of this system using two different methods:

   (a) Integrate

   $$\frac{1}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right) \quad \text{(SI units)}$$
   $$\text{or}$$
   $$\frac{1}{8\pi}\left(E^2 + B^2\right) \quad \text{(Gaussian units)}$$

   over the volume.

   (b) Integrate $\rho\,\Phi/2$ (SI or Gaussian units), where Φ is the electric potential, over the volume.

2. 20 points   (a) Prove the electromagnetic mean value theorem (MVT):

For charge-free space V, the value of the electrostatic potential $\Phi(\vec{x})$ at any point in V is equal to the average of the potential over the surface of any sphere centered on that point.

(b) Use the MVT to provide a proof of the uniqueness theorem of electrodynamics:

The solution to the Poisson equation

$$\nabla^2 \Phi(\vec{x}) = -\rho(\vec{x})/\epsilon_0 \quad \text{(SI units)}$$

$$\nabla^2 \Phi(\vec{x}) = -4\pi\rho(\vec{x}) \quad \text{(Gaussian units)}$$

for the electrostatic potential inside a volume V, subject to the surface boundary condition $\Phi(\vec{x})|_S = f(\vec{x})$, is unique.

3. 20 points A rotating charged sphere with radius a and angular frequency ω has a non-uniform surface charge density given by $\sigma(\theta) = \sigma_0 \cos\theta$, where θ is the polar angle in spherical coordinates. The rotating sphere generates a surface current

$$\vec{K}(\theta) = \omega a \sigma(\theta) \sin\theta \, \hat{e}_\phi,$$

which gives rise to a magnetic vector potential $\vec{A}$. Using spherical coordinates and the Coulomb expansion

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell,m} \frac{4\pi}{2\ell + 1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi),$$

where $Y_{\ell m}(\theta, \phi)$ are the spherical harmonics and $r_<$ ($r_>$) is the lesser (greater) of r and r', show that the vector potential (Coulomb gauge) is

$$\vec{A} = \frac{\mu_0 \sigma_0 a^3 \omega}{5} \frac{r_<^2}{r_>^3} \sin\theta \cos\theta \, \hat{e}_\phi \quad \text{(SI units)}$$

or

$$\vec{A} = \frac{4\pi \sigma_0 a^3 \omega}{5c} \frac{r_<^2}{r_>^3} \sin\theta \cos\theta \, \hat{e}_\phi \quad \text{(Gaussian units)}.$$

Here, $r_<$ ($r_>$) is the lesser (greater) of r and a, and $\hat{e}_\phi$ is the azimuthal unit vector.

(You do not need to calculate the magnetic field, $\vec{B} = \vec{\nabla} \times \vec{A}$.)

4. 20 points A cylinder of radius R is located along the z axis. The bottom and top of the cylinder on the $z = 0$ and $z = L$ planes have $\Phi = 0$. The sides of the cylinder at $r = R$ have:

$$\Phi(r = R, \phi, z) = \begin{cases} V_0 & \text{for } 0 \leq \phi < \pi, \text{ all } z \\ -V_0 & \text{for } \pi \leq \phi < 2\pi, \text{ all } z. \end{cases}$$

Find $\Phi(r, \phi, z)$ everywhere inside of the cylinder. You may begin with the following general form involving modified Bessel functions K_m and I_m:

$$\Phi(r, \phi, z) = \sum_{m,n} (A_m K_m(k_n r) + B_m I_m(k_n r)) \times$$

$$(C_m \sin m\phi + D_m \cos m\phi) \times (E_n \sin k_n z + F_n \cos k_n z).$$

5. 20 points The Maxwell Stress tensor T_{ij} gives the force per unit area (or stress) acting on a surface due to electromagnetic fields. The elements of the tensor are given by

$$T_{ij} \equiv \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right) \text{ (SI units)}$$

or

$$T_{ij} \equiv \frac{1}{4\pi} \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right) + \frac{1}{4\pi} \left(B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right) \text{ (Gaussian units)}.$$

(a) Given a monochromatic (all photons have the same angular frequency ω) plane wave traveling in the z direction and linearly polarized in the x direction, write down the components of the electric field of the plane wave. Take the amplitude of the electric field associated with the plane wave to be E_0.

(b) Use Maxwell's equations to derive the components of the magnetic field.

(c) Make a spatial sketch of the electric and magnetic fields associated with this plane wave.

(d) Find all elements of the Maxwell stress tensor for this monochromatic plane wave and interpret the answer. Note that T_{ij} represents the momentum flux density. How is the momentum flux density related to the energy density in this case?

Statistical Mechanics
Preliminary Exam, Spring 2023

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Grading: There are **5** questions on this exam. Each problem is worth 20 points.

1. [20 points] At short lengths, a spring has a free energy A given by

$$A/M = \frac{1}{2}kx^2$$

where M is the mass of the spring, k is an effective spring constant with appropriate units, $x = L/M$ is the length per unit mass, and $A = U - TS$. If the spring is stretched beyond its elastic limit the free energy of the spring is then given by

$$A/M = \frac{1}{2}h(x - x_0)^2 + C.$$

The constants k, h, x_0 and C are all independent of x but depend upon T, and $k > h$, $C > 0$ and $x_0 > 0$ for all T. Given that

$$dA = -S\,dT + f\,dL + \mu\,dM,$$

For both short and long spring lengths:

(a) Determine $f = f(T, x)$, the tension of the spring

(b) Determine the chemical potential $\mu = (\partial A/\partial M)_{T,L}$.

(c) Show that $\mu = A/M - fx$.

2. 20 points Consider an arbitrary reversible heat engine which operates between two finite "reservoirs", each of which has the same finite temperature-independent heat capacity C. The reservoirs have initial temperatures T_1 and T_2, where $T_2 > T_1$, and the engine operates until both reservoirs have the same final temperature T_3.

  (a) Calculate the changes in the entropy for each of the hot and cold reservoirs.

  (b) Calculate the heat flow for each of the hot and cold reservoirs.

  (c) Give an argument which shows that $T_3 \geq \sqrt{T_1 T_2}$.

  (d) What is the maximum amount of work obtainable from the engine?

3. 20 points Consider a system of N non-interacting and indistinguishable particles, each possessing a magnetic dipole moment $\pm\mu$ in an external magnetic field B. The average volume per particle (V/N) is so much larger than the quantum volume that the quantum effect can be neglected. The energy of such a particle is given by

$$E = \frac{p^2}{2m} \mp \mu B$$

(a) Calculate the single particle partition function, starting from $Q_1 = \sum e^{-\beta E}$, where $\beta = 1/k_B T$.

(b) Calculate the energy of the system of N particles and obtain the limiting values as $k_B T \gg \mu B$ as well as $k_B T \ll \mu B$.

4. 20 points Take the total energy (potential plus thermal) of the Sun to be given by the expression

$$E \approx -\frac{3}{10-2n} \frac{GM^2}{R}$$

where M and R are the mass and radius, respectively. The Sun is fairly well represented by $n = 3$. Suppose that the energy generation in the Sun were suddenly turned off and the Sun began to slowly contract. During this contraction its mass, M, would remain constant and, to a fair approximation, its surface temperature would also remain constant at 5800 K. Assume that the total energy of the Sun is always given by the above expression, even as R gets smaller. Write down the differential equation relating the power radiated at the Sun's surface with the change in its total energy (using the above expression). Integrate this equation to find the time (in years) for the Sun to shrink to $1/2$ its present radius. Note: the luminosity (energy emitted per second) of a star is given by

$$L = 4\pi R^2 \sigma T^4$$

where σ is the Stephan-Boltzmann constant.

5. 20 points The probability distribution of the speed of molecules with mass m at temperature T can be written as

$$F(v) = \sqrt{\frac{m}{2\pi k_B T}} \exp\left(\frac{-mv^2}{2k_B T}\right)$$

The Doppler formula for the observed frequency f from a source moving with velocity v along the line of sight to the observer is

$$f = f_0\left(1 + \frac{v}{c}\right)$$

where f_0 is the frequency radiated when the source is at rest and c is the speed of light.

(a) What is the distribution in the observed frequency f of a particular spectrum line with frequency f_0, $F(f - f_0)$, radiated from a gas at temperature T?

(b) Find the variance $\langle (f - f_0)^2 \rangle$ of this radiated frequency. The square root of the variance is called the "breadth" of the line.

(c) Note that although the distribution in speeds extends to $v = \infty$, the probability of finding a molecule with a speed $> 10^5$ m/s in most astrophysical sources is vanishingly small. Comment on the importance of this fact to the calculations in parts (a) and (b).

(d) Atomic hydrogen H^1 and atomic oxygen O^{16} are both present in a hot gas. How much broader will the hydrogen lines be, compared to the oxygen lines of roughly the same frequency?